9. CIRCLE

1. INTRODUCTION

Definition: The locus of a point which moves in a plane such that its distance from a fixed point in that plane always remains the same (i.e., constant) is known as a circle.

The fixed point is called the centre of the circle and the distance between the fixed point and moving point is called the radius of the circle.

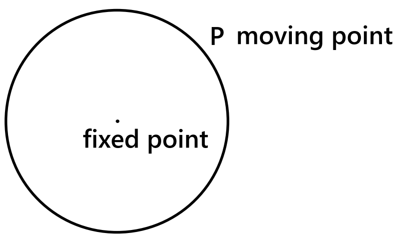

Figure 9.1

2. DIFFERENT FORM OF EQUATION OF CIRCLE

2.1 General Form

The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ where g, f, and c are constants.

(a) Centre of the circle ≡ $(-g, -f)$. i.e., $(-\frac{1}{2}$ coefficient of x, $-\frac{1}{2}$ coefficient of y).

(b) Radius of the circle = $\sqrt{g^2 + f^2 - c}$.

Nature of the circle:

(a) If $g^2 + f^2 - c > 0$, then the radius of the circle will be real. Hence, it is possible to draw a circle on a plane.

(b) If $g^2 + f^2 - c = 0$, then the radius of the circle will be zero. Such a circle is known as point circle.

(c) If $g^2 + f^2 - c < 0$, then the radius $\sqrt{g^2 + f^2 - c}$ of the circle will be an imaginary. Hence, it is not possible to draw a circle.

The condition for the general second degree equation to represent a circle:

The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle iff

(a) $a = b \neq 0$ i.e. the coefficient of x^2 = the coefficient of $y^2 \neq 0$.

(b) $h = 0$ i.e. the coefficient of xy is 0.

(c) $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$, it implies that the general equation is non degenerate (i.e. equation cannot be written into two linear factors)

(d) $g^2 + f^2 - c \geq 0$

PLANCESS CONCEPTS

- The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2gx + 2fy + c = 0 \text{ and } \begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- Degeneracy condition depends on determinant of 3x3 matrix and the type of conic depends on determinant of 2x2 matrix.
- Also the equation can be taken as intersection of $z = ax^2 + 2hxy + by^2$ and the plane $z = -(2gx + 2fy + c)$

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2.2 Standard Form

The equation of circle with center (0,0) and radius 'a' is $x^2 + y^2 = a^2$.

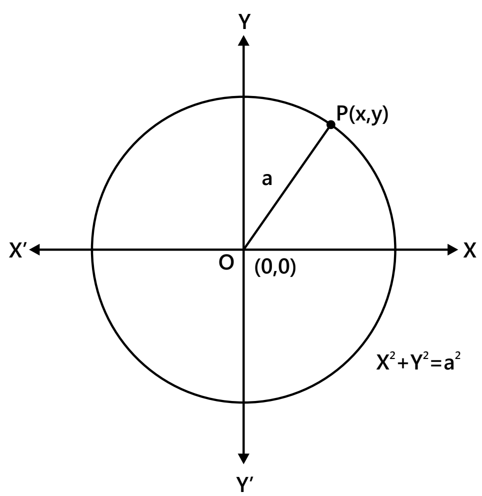

Figure 9.2: Standard Form

2.3 Central Form

The equation of the circle with centre (h, k) and radius 'r' is $(x - y)^2 + (y - k)^2 = r^2$.

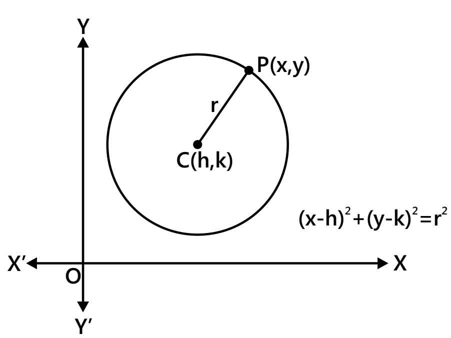

Figure 9.3: Central form

2.4 Diametric Form

Circle on a given diameter: The equation of the circle with (x_1, y_1) and (x_2, y_2) as the end points of the diameter is given by

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Centre $\equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

and, Radius = $\sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$.

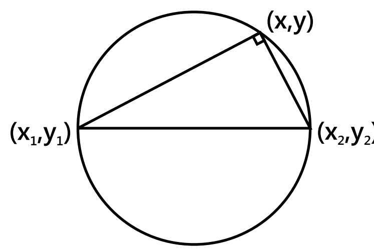

Figure 9.4: Diametric Form

2.5 Parametric Form

For $x^2 + y^2 = r^2$, parametric co-ordinates of any point on the circle is given by $(r \cos\theta, r \sin\theta)$, $(0 \le \theta < 2\pi)$.

(a) The parametric co-ordinates of a point on the circle $(x - h)^2 + (y - k)^2 = r^2$ is given by $(h + r\cos\theta, k + r\sin\theta)$, $(0 \le \theta < 2\pi)$.

(b) The parametric co-ordinates of any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + r\cos\theta$ and $y = -f + r\sin\theta$, (where $r = \sqrt{g^2 + f^2 - c}$, and $0 \le \theta < 2\pi$)

2.6 Equation of Circle under Special Conditions

(a) The equation of the circle through three points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

The concept of family of circles can be used to derive this form.

Taking any two of the given three points as extremities of diameter, we get the equation of a circle S = 0 and the equation of straight line passing through these two points L = 0. Then the equation of circle passing through the intersection of circle and line is $S + \lambda L = 0$, where λ is a parameter. The value of λ can be found by substituting the third point in the equation as it also lies on the circle.

(b) Equation of circle circumscribing the triangle formed by the lines $a_r x + b_r y + c_r = 0$ where r = 1, 2, 3 is

$$\begin{vmatrix} \frac{a_1^2 + b_1^2}{a_1 x + b_1 y + c_1} & a_1 & b_1 \\ \frac{a_2^2 + b_2^2}{a_2 x + b_2 y + c_2} & a_2 & b_2 \\ \frac{a_3^2 + b_3^2}{a_3 x + b_3 y + c_3} & a_3 & b_3 \end{vmatrix} = 0$$

PLANCESS CONCEPTS

Whenever the problem seems to be very complicated using formulas and geometrical approach, then try to apply trigonometric approach as well. Like the given circle may be in circle or ex-circle of some triangle. May be using properties of triangle we can solve it.

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Illustration 1: Find the equation of the circle which passes through the point of intersection of the lines $x - 4y - 1 = 0$ and $4x + y - 21 = 0$ and whose centre is $(2, -3)$. **(JEE MAIN)**

Sol: By solving given equation of lines simultaneously we will get point of intersection of lines i.e. P. Therefore using distance formula we will get radius of circle and using centre and radius we will get required equation.

Let P be the point of intersection of the lines

$$x - 4y - 1 = 0 \quad \text{.... (i)}$$

and $$4x + y - 21 = 0 \quad \text{.... (ii)}$$

From (i) and (ii), we get $x = 5, y = 1$. So, coordinates of P are $(5,1)$.Let $C(2,-3)$ be the centre of the circle.

Since the circle passes though P, therefore

CP= radius $\Rightarrow \sqrt{(5-2)^2 + (1+3)^2}$ = radius $\Rightarrow$ radius $= 5$

Hence the equation of the required circle is $(x - 2)^2 + (y + 3)^2 = 25$.

Illustration 2: Find the equation of a circle of radius 10 whose centre lies on x-axis and passes through the point (4, 6). **(JEE MAIN)**

Sol: Consider coordinates of centre of circle as (a, 0). Now by using distance formula, we can calculate the value of 'a' and then by using central form, we get required equation.

Here centre C lies on x-axis, and the circle passes through A (4,6).

Let C be (a,0)

$\therefore$ CA = radius

$\Rightarrow$ CA =10 $\quad \Rightarrow \sqrt{(a-4)^2 + (0-6)^2} = 10$

$\Rightarrow (a - 4)^2 + 36 = 100 \quad \Rightarrow \quad (a - 4)^2 = 64$

$\Rightarrow a - 4 = \pm 8 \quad \Rightarrow \quad a = 12$ or $a = -4$.

Thus, the coordinates of the centre are (12, 0) or $(-4,0)$.

Hence, the equations of the required circles are

$(x-12)^2 + (y-0)^2 = 10^2$ and $(x+4)^2 + (y-0)^2 = 10^2$

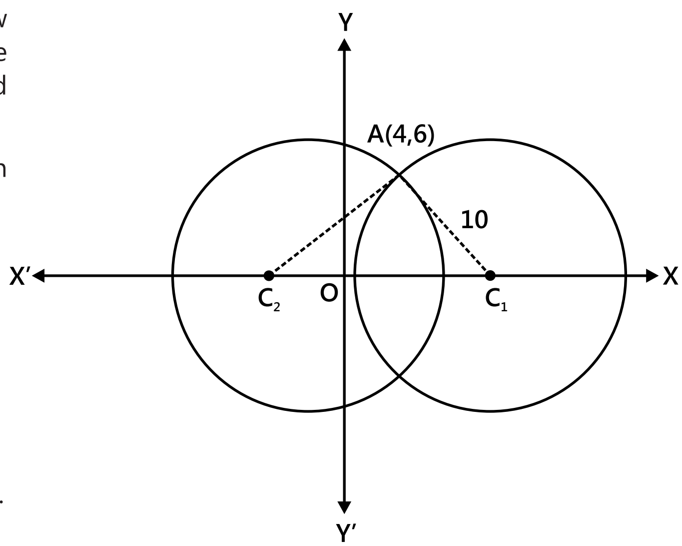


Figure 9.5

Illustration 3: Find the equation of the circle concentric with the circle $x^2 + y^2 + 4x + 6y + 11 = 0$ and passing through the point $(5,4)$. **(JEE MAIN)**

Sol: Since both circle are concentric therefore their centre should be same. Hence equation of a required circle can be written as $x^2 + y^2 + 4x + 6y +$ (constant term) = 0.

Let the equation of the concentric circle be

$$x^2 + y^2 + 4x + 6y + k = 0$$

Since the point (5,4) lies on this circle,

$\therefore (5)^2 + (4)^2 + 4(5) + 6(4) + k = 0$

$\Rightarrow 25 + 16 + 20 + 24 + k = 0 \quad \Rightarrow k = -85$

Therefore, the equation of the required circle is

$$x^2 + y^2 + 4x + 6y - 85 = 0$$

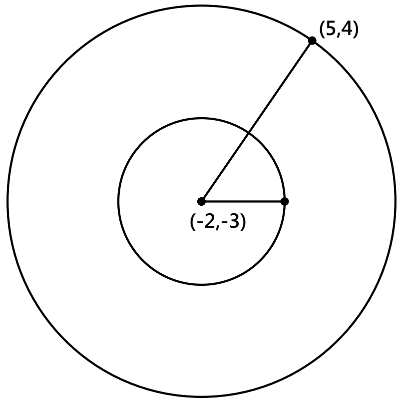


Figure 9.6

Illustration 4: Find the equation of a circle passing through the origin and making intercepts 4 and 3 on the y and x axis respectively. **(JEE MAIN)**

Sol: By observing the problem we conclude that given intercepts are end points of diameter of this circle. Therefore by using diametric form we can obtain the equation of circle.

Let the intercepts be OP = 4, OQ = 5

$\therefore$ The co-ordinates of P and Q are (0, 4) and (3, 0) respectively.

Since POQ = 90°, hence PQ is a diameter.

$\therefore$ The required equation of the circle is

$(x-3)(x-0)+(y-0)(y-4)=0 \Rightarrow \quad x^2+y^2-3x-4y=0$

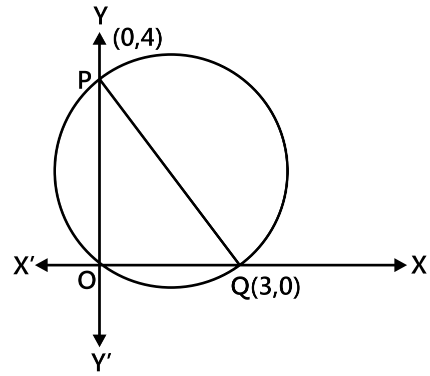


Figure 9.7

Illustration 5: Find the equation of the circle which is circumscribed about the triangle whose vertices $(-2,3)$, $(5,2)$ and $(6,-1)$. **(JEE ADVANCED)**

Sol: Consider (a, b) as the centre of circle and r as the radius. As circle passes from given vertices, therefore their distance from the centre are same. Therefore by using distance formula, we will get the value of a, b and r.

Since the circle passes through the points $(-2,3)$, $(5,2)$ and $(6,-1)$.

$\therefore \quad (-2-a)^2+(3-b)^2=r^2 \Rightarrow a^2+4a+4+b^2-6b+9=r^2$ .... (i)

$(5-a)^2+(2-b)^2=r^2 \Rightarrow a^2-10a+25+b^2-4b+4=r^2$ .... (ii)

$(6-a)^2+(-1-b)^2=r^2 \Rightarrow a^2-12a+36+b^2+2b+1=r^2$ .... (iii)

Subtracting (ii) from (i), we have

$14a-21-2b+5=0 \quad$ i.e., $14a-2b=16$ .... (iv)

Subtracting (iii) from (ii), we get

$2a-11-6b+3=0 \quad \Rightarrow 2a-6b=8$ .... (v)

Solving (iv) and (v), we get

$a=1$ and $b=-1$

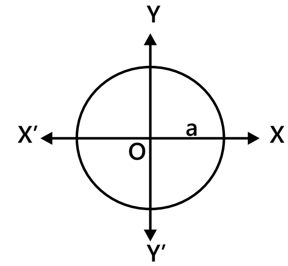
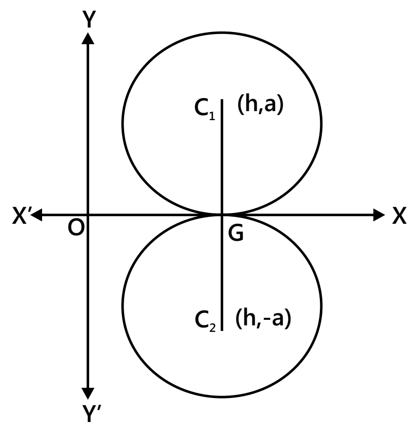
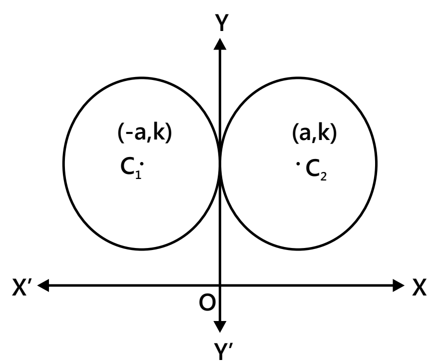
Putting the values of $a=1$ and $b=-1$ in (i), we get

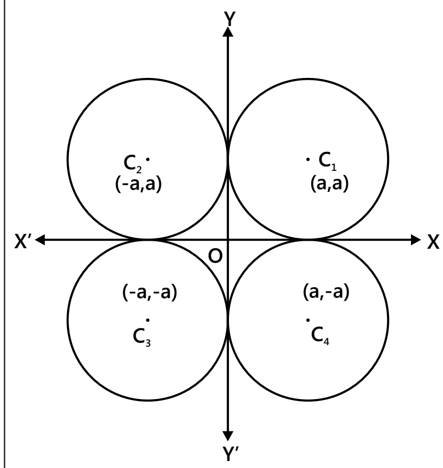
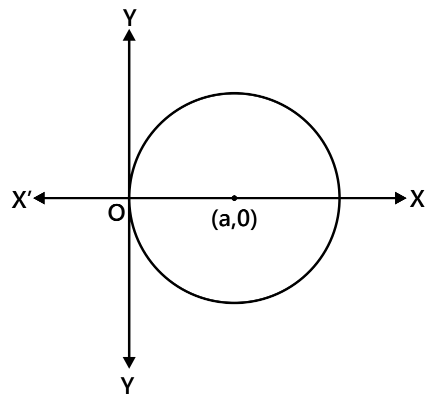
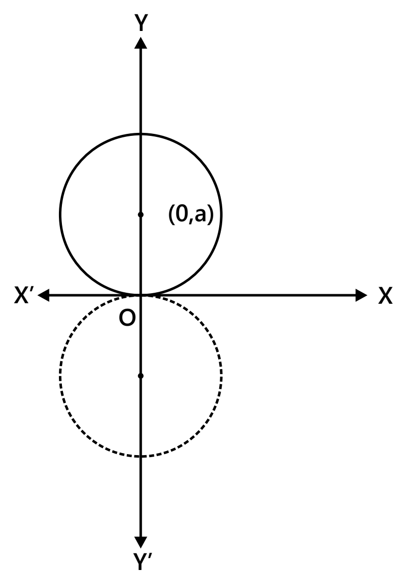
$1+4+4+1+6+9=r^2 \quad \Rightarrow 25=r^2 \quad \Rightarrow r=5$

Thus, the required equation of the circle is

$(x-1)^2+(y+1)^2=25 \quad \Rightarrow x^2+2x+1+y^2+1+2y=25 \Rightarrow x^2+y^2-2x+2y-23=0$

3. EQUATION OF CIRCLE IN SOME SPECIAL CASES

Equation	Centre/Radius	Properties	Figures
(a) $x^2 + y^2 = a^2$	$(0, 0)$; a	When the centre of the circle coincides with the origin center = (0,0)	 Figure 9.8
(b) $(x - h)^2 + (y \pm a)^2 = a^2$	$(h, \pm a)$; a	Touches x axis only, y coordinate of centre = $\pm a$	 Figure 9.9
(c) $(x \pm a)^2 + (y - k)^2 = a^2$	$(\pm a, k)$; a	Touches y–axis only, x coordinate of centre = $\pm a$	 Figure 9.10

Equation	Centre/Radius	Properties	Figures
(d) $(x \pm a)^2 + (y \pm a)^2 = a^2$	$(\pm a, \pm a)$; a	Touches both the axes depending on the quadrant center = $(\pm a, \pm a)$	 **Figure 9.11**
(e) $x^2 + y^2 - 2ax = 0$	C (a, 0) ; a	When the circle passes through the origin and centre lies on x axis	 **Figure 9.12**
(f) $x^2 + y^2 - 2ay = 0$	C (0, a); a	When the circle passes through the origin and centre lies on y axis.	 **Figure 9.13**

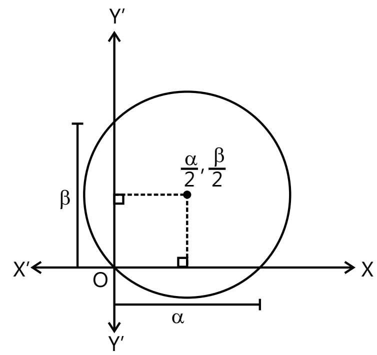
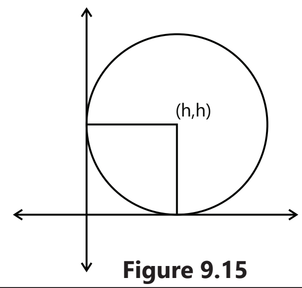
Equation	Centre/Radius	Properties	Figures
(g) $x^2 + y^2 - \alpha x - \beta y = 0$	$\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$; $\frac{1}{2}\sqrt{\alpha^2 + \beta^2}$	Passes through (0, 0) and has intercepts α and β on x & y axes respectively.	 **Figure 9.14**
(h) $(x-h)^2 + (y-h)^2 = h^2$ or $x^2 + y^2 - 2hx - 2hy + h^2 = 0$	(h, h); h	When circle touches both the axes	 **Figure 9.15**

Illustration 6: Find the equation of the circle which passes through two points on the x–axis which are at distances 12 from the origin and whose radius is 13.

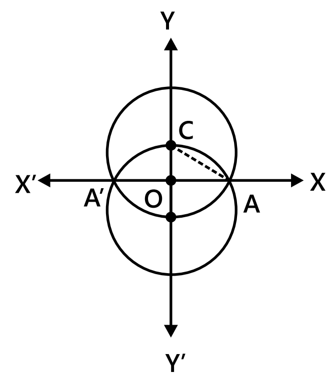


Figure 9.16

Sol: There are two circles which passes through two points A and A′ on x–axis which are at a distance 12 from the origin and whose radius is 13. The centre of these circles lie on y–axis (perpendicular bisector of chord AA′)

In ΔAOC, $AC^2 = OA^2 + OC^2$

$\Rightarrow 13^2 = 12^2 + OC^2 \quad \Rightarrow OC = 5$

So the coordinates of the centre of the required circles are (0, 5) and C′ (0, –5). Hence the equations of the required circles are

$(x-0)^2 + (y \pm 5)^2 = 13^2 \Rightarrow x^2 + y^2 \pm 10y - 144 = 0.$

4. INTERCEPTS ON THE AXES

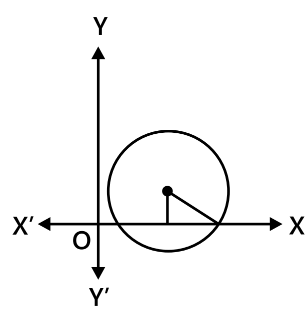


Figure 9.17: Intercept made by circle on x-axis

The lengths of intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on X and Y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.

Therefore,

(a) The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the x–axis in real and distinct points, touches or does not meet in real points according as $g^2 > c$- Distinct points

$g^2 = c$ – Touches

$g^2 < c$ – Does not meet

(b) Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the y–axis in real and distinct points, touches or does not meet in real points according as $f^2 >, =$ or $< c$.

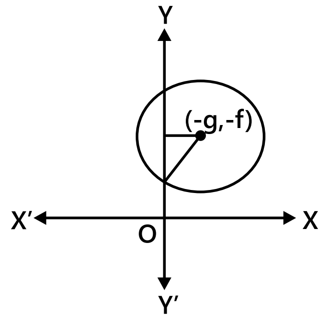


Figure 9.18: Intercept made by circle on y-axis

Illustration 7: Find the equation to the circle which touches the positive axis of y at a distance 4 from the origin and cuts off an intercept of 6 from the axis of x. **(JEE MAIN)**

Sol: As circle touches Y axis therefore Y coordinate of centre of circle is 4 so by using formula of intercept we will get the value of X coordinate of centre of circle and c.

Consider a circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

This meets the axis of y in points given by $y^2 + 2fy + c = 0$

The roots of this equation must be each equal to 4, so that it must be equivalent to $(y - 4)^2 = 0 \Rightarrow 2f = -8$ & $c = 16$

$\because$ Intercept made on the x-axis = 6

$\Rightarrow 6 = 2\sqrt{g^2 - 16} \quad \Rightarrow g = \pm 5.$

Hence, the required equation is $x^2 + y^2 \pm 10x - 8y + 16 = 0$.

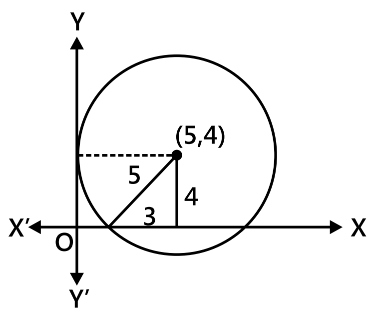


Figure 9.19

5. POSITION OF A POINT W.R.T A CIRCLE

(a) If CP < radius, then the point P lies inside the circle. (Refer Fig. 9.20 (i))

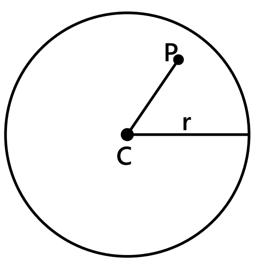


Figure 9.20 (i)

(b) If CP = radius, then the point P lies on the circumference.

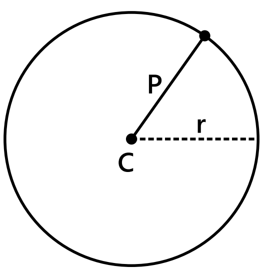


Figure 9.20 (ii)

(c) CP > radius, then the point P lies outside the circle.

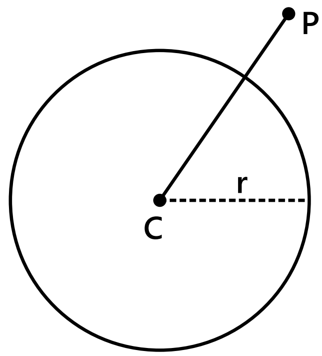


Figure 9.20 (iii)

Hence, any point (x, y) lies outside, on or inside if

$\sqrt{(x_1+g)^2+(y_1+f)^2}$ > = < $\sqrt{g^2+f^2-c}$ $\Rightarrow (x_1 + g^2) + (y_1 + f)^2 >$

$= < (g^2 + f^2 - c)$

$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > = < 0$

Or, $S_1 > = < 0$ where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Therefore, a point (x_1, y_1) lies outside, on or inside a circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative.

5.1 Power of a Point w.r.t. a circle

Let P (x_1, y_1) be a point and a secant (a line which cuts the curve in two point) PAB is drawn.

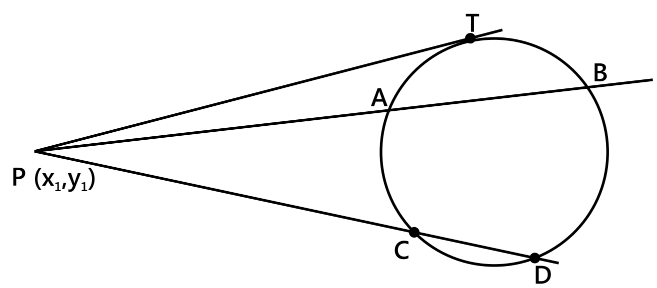


Figure 9.21

The power of $P(x_1, y_1)$ w.r.t. $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to PA.PB which is S_1, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Power remains constant for the circle i.e. independent of A and B

$PA \times PB = PC \times PD = PT^2$ = square of the length of a tangent

5.2 The Least and Greatest distance of a Point from a Circle

Let S = 0 be a circle and P (x_1, y_1) be a point. If the diameter of the circle through P intersect the circle at Q and R,

then QP = $|PC - r|$ = least distance; and

PR = PC + r = greatest distance

where 'r' is the radius and C is the centre of the circle.

(Refer Fig. 9.23)

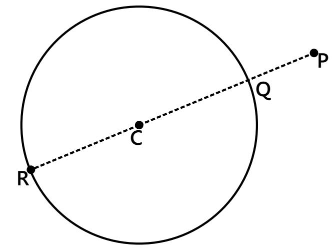


Figure 9.22

Illustration 8: The coordinates of the point on the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ farthest from the origin are

(JEE MAIN)

(A) $\left(2+\frac{8}{\sqrt{5}}, 1+\frac{4}{\sqrt{5}}\right)$ (B) $\left(1+\frac{4}{\sqrt{5}}, 2+\frac{8}{\sqrt{5}}\right)$ (C) $\left(1+\frac{8}{\sqrt{5}}, 2+\frac{4}{\sqrt{5}}\right)$ (D) None of these

Sol: (B) The required point lies on the normal to circle through the origin, i.e. on the line 2x = y. Therefore by substituting y = 2x in above equation of circle we will get coordinates of required point.

$x^2 + 4x^2 - 2x - 8x - 11 = 0 \Rightarrow 5x^2 - 10x - 11 = 0 \Rightarrow x = 1 \pm \frac{4}{\sqrt{5}}$ and $y = 2\left(1 \pm \frac{4}{\sqrt{5}}\right)$

and the required point farthest from the origin is $\left(1+\frac{4}{\sqrt{5}}, 2+\frac{8}{\sqrt{5}}\right)$.

Illustration 9: The point (1, 3) is inside the circle S whose equation is of the form $x^2 + y^2 - 6x - 10y + k = 0$, k being an arbitrary constant. Find the possible values of k if the circle S neither touches the axes nor cuts them.

(JEE ADVANCED)

Sol: As (1, 3) lies inside the circle S therefore $S_1 < 0$ and it does not touches x and y axes. On the basis of this we can solve the problem and will get range of k.

$1^2 + 3^2 - 6\times1 - 10\times3 + k < 0$; $\therefore k < 26$ ...(i)

Solving y = 0 and $x^2 + y^2 - 6x - 10y + k = 0$, we get $x^2 - 6x + k = 0$

Since the circle S does not intersect with the x-axis,

$\Rightarrow$ discriminant < 0 i.e., $36 - 4k < 0$ $\Rightarrow k > 9$ ...(ii)

Solving x = 0 and $x^2 + y^2 - 6x - 10y + k = 0$, we get $y^2 - 10y + k = 0$

Since the circle S does not intersect with the y-axis,

$\Rightarrow$ discriminant < 0 i.e., $100 - 4k < 0$ $\Rightarrow k > 25$ ...(iii)

From (i), (ii) and (iii), we get $25 < k < 26$, i.e., $k \in (25, 26)$.

6. LINE AND A CIRCLE

The length of the intercept cut off from the line y = mx + c by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$.

(a) If $a^2(1 + m^2) > c^2$, or, $|c| < a\sqrt{1+m^2}$

i.e., the line will intersects the circle at two real and different points.

(b) If $a^2(1 + m^2) = c^2$, or, $|c| = a\sqrt{1+m^2}$

i.e., the line will touch the circle at only one point i.e. the line will be a tangent.

(c) If $a^2(1 + m^2) < c^2$, or, $|c| > a\sqrt{1+m^2}$

i.e., the line will meet the circle at two imaginary points.

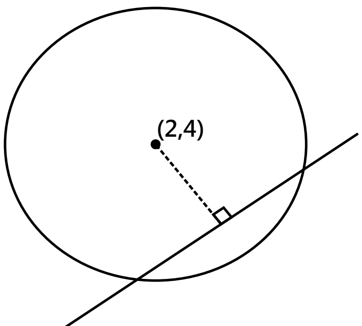


Figure 9.23

Illustration 10: Show that the line $3x - 4y - c = 0$ will meet the circle having centre at (2, 4) and the radius 5 in real and distinct points if $-40 < c < 20$.

(JEE MAIN)

Sol: Since the line cuts the circle so length of perpendicular from centre of circle upon line is less than the radius of circle.

$\left|\frac{3\times2-4\times4-c}{\sqrt{9+16}}\right| < 6 \quad \Rightarrow |10 + c| < 30$

$\Rightarrow \quad -30 < 10 + c < 30 \quad \Rightarrow -40 < c < 20$

Illustration 11: If $4l^2 - 5m^2 + 6l + 1 = 0$ then show that the line $lx + my + 1 = 0$ touches a fixed circle. Find the radius and centre of the circle. **(JEE ADVANCED)**

Sol: If line touches the circle then perpendicular distance from centre of circle to the line is equal to the radius of circle so by using distance formula of point to line we will get one equation and other is given $4l^2 - 5m^2 + 6l + 1 = 0$. hence by solving these two equation we will get required answer.

Let the circle be $(x - \alpha)^2 + (y - \beta)^2 = a^2$

The line $lx + my + 1 = 0$ touches the circle if

$a = \left|\frac{l\alpha + m\beta + 1}{\sqrt{l^2 + m^2}}\right| \quad$ or $\quad a^2(l^2 + m^2) = (l\alpha + m\beta + 1)^2$

or $\quad (a^2 - \alpha^2)\, l^2 + (a^2 - \beta^2)\, m^2 - 2\, l\beta\alpha\, m - 2\, l\, \alpha - 2m\beta - 1 = 0$ ... (i)

But $\quad 4l^2 - 5\, m^2 + 6l + 1 = 0$ ... (ii)

It is possible to find α,β, a if (i) and (ii) are identical.

The condition is

$$\frac{a^2 - \alpha^2}{4} = \frac{a^2 - \beta^2}{-5} = \frac{-2\alpha}{6} = \frac{2\beta}{0} = \frac{-1}{1} \quad \text{... (iii)}$$

$\therefore\ \beta = 0,\ \alpha = 3$ and $\frac{a^2 - \alpha^2}{4} = -1$ which implies $a^2 - 3^2 = -4$, i.e., $a = \sqrt{5}$ .Also $\alpha = 3,\ \beta = 0,\ a = \sqrt{5}$ satisfies equation (iii) and hence the line touches the fixed circle $(x - 3)^2 + (y - 0)^2 = \left(\sqrt{5}\right)^2$

or $x^2 + y^2 - 6x + 4 = 0$, whose centre $= (\alpha, \beta) = (3, 0)$ and radius $= a = \sqrt{5}$.

Illustration 12: Find equation of a line with slope gradient 1 and such that $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal length on it ? **(JEE ADVANCED)**

Sol: As given slope of line is 1, therefore its equation will be $y = x + c$ Hence by using perpendicular distance formula we will get distance of line from centre of respective circle and then by using Pythagoras we can obtain length of intercepts made by line to these circles and which are equal. Therefore we can obtain value of c and required equation of circle.

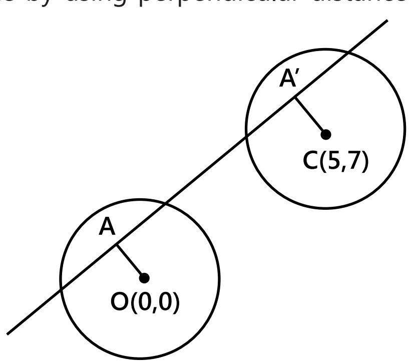


Figure: 9.24

Let 2ℓ be the length of the intercept made by the two circle.

For $x^2 + y^2 = 4$, centre ≡ (0,0) and radius = 2, and

For $x^2 + y^2 - 10x - 14y + 65 = 0$, centre ≡ (5,7) and radius = 3.

$\therefore\ OA = \left|\frac{c}{\sqrt{2}}\right| \quad$ and $CA' = \left|\frac{5 - 7 + c}{\sqrt{2}}\right| \Rightarrow CA' = \left|\frac{c-2}{\sqrt{2}}\right|$

Now, from the diagram we get

$4 - OA^2 = \ell^2$ ...(i)

and $9 - CA'^2 = \ell^2$ ...(ii)

$\Rightarrow 4 - \frac{c^2}{2} = 9 - \left(\frac{c-2}{2}\right)^2$ [from (i) and (ii)]

$\Rightarrow c = -\frac{3}{2}$

The equation of line is 2x - 2y = 3

Illustration 13: Find the values of α for which the point $(2\alpha, \alpha + 1)$ is an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$ made by the chord whose equation is $x - y + 1 = 0$. **(JEE ADVANCED)**

Sol: As point $(2\alpha, \alpha + 1)$ lies inside the circle S, therefore $S_1 < 0$. Hence by substituting the point in the equation, we will get the range of α and as it lies in larger segment made by line $x - y + 1 = 0$. The centre of circle i.e. (1, 1) and $(2\alpha, \alpha + 1)$ will have the same sign.

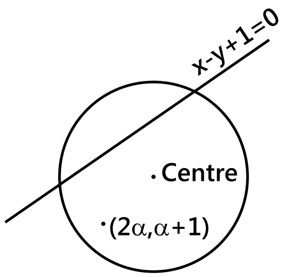


Figure 9.25

$\therefore (2\alpha)^2 + (\alpha + 1)^2 - 2 \times 2\alpha - 2(\alpha + 1) - 8 < 0$

$\Rightarrow 5\alpha^2 - 4\alpha - 9 < 0$ or $(5\alpha - 9)(\alpha + 1) < 0$

$\Rightarrow -1 < \alpha < 9/5$ ... (i)

Also, as the point lies in the larger segment, the centre (1, 1) and the point $(2\alpha, \alpha + 1)$ must be on the same side of the line $x - y + 1 = 0$.

Clearly, $1 - 1 + 1 > 0$; So, $2\alpha - (\alpha + 1) + 1 > 0$;

$\therefore \alpha > 0$ ... (ii)

$\therefore$ The set of values of α satisfying (i) and (ii) is $\left(0, \frac{9}{5}\right)$.

7. TANGENTS

7.1. Point Form

(a) The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 - a^2 = 0$.

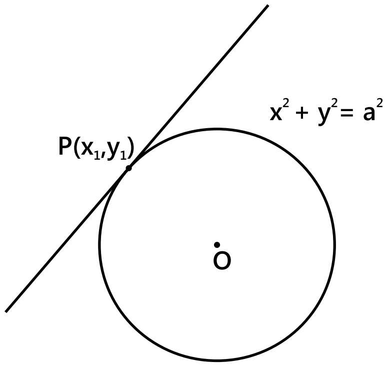


Figure 9.26

(b) The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

7.2. Parametric Form

Since parametric co-ordinates of a point on the circle $x^2 + y^2 = a^2$ is $(a \cos \theta, a \sin \theta)$, then equation of tangent at $(a \cos \theta, a \sin \theta)$ is $x\, a \cos \theta + y\, a \sin \theta = a^2$ or, $x \cos \theta + y \sin \theta = a$.

7.3. Condition for Tangency

A line L = 0 touches the circle S = 0. If length of perpendicular (p) drawn from the centre of the circle to the line is equal to radius of the circle i.e. p = r. This is the condition of tangency for the line L = 0. Circle $x^2 + y^2 = a^2$ will touches the line y = mx + c if $c = \pm a\sqrt{1+m^2}$

Illustration 14: For what value of c will the line y = 2x + c be a tangent to the circle $x^2 + y^2 = 5$. **(JEE MAIN)**

Sol: The equation of the tangent to the circle $x^2 + y^2 = a^2$ in slope form is $y = mx + a\sqrt{1+m^2}$.

On comparison, we get $a = \sqrt{5}$ and m = 2.

$\therefore c = \pm\ \sqrt{5}\times\sqrt{1+2^2}$ $\qquad \Rightarrow c = \pm\ 5$

The required equation is $y = 2x \pm 5$

7.4. Slope Form

The straight line y = mx + c touches the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Therefore, the equation of the tangent in the slope form is $y = mx \pm a\sqrt{1+m^2}$ and the point of contact is $\left(\dfrac{\mp ma}{\sqrt{1+m^2}}, \dfrac{\pm a}{\sqrt{1+m^2}}\right)$.

7.5 Length of Tangent

The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is $PT_1 = PT_2 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$\therefore$ Length of Tangent = $\sqrt{S_1}$

Note:

(i) PT^2 is called the power of the point with respect to a given circle, where PT is the tangent from a point P to a given circle.

(ii) Area of quadrilateral $PT_1C\,T_2$ = 2 × (Area of triangle PT_1C), and

(iii) The angle between tangents PT_1 and PT_2 is equal to $2\tan^{-1}\left(\dfrac{r}{\sqrt{S_1}}\right)$.

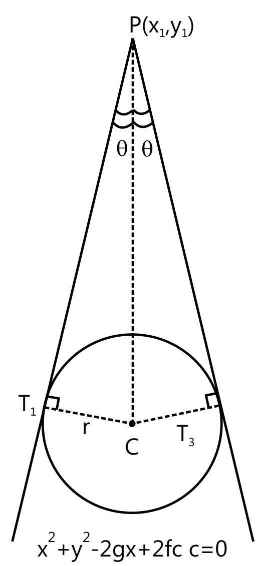


Figure 9.27

Illustration 15: If OA and OB are tangents from the origin O, to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, c > 0 and C is the centre of the circle, then area of the quadrilateral OACB is **(JEE MAIN)**

(A) $\dfrac{1}{2}\sqrt{c\left(g^2+f^2-c\right)}$ (B) $\sqrt{c\left(g^2+f^2-c\right)}$

(C) $c\sqrt{g^2+f^2-c}$ (D) $\sqrt{\dfrac{g^2+f^2-c}{c}}$

Sol: (B) As we know quadrilateral OACB is formed by two right angle triangle OAC and triangle OBC. Line OA and OB are tangent to the circle from common point O. Therefore OA = OB and (AC = CB) radius of circle, hence both triangle are equal. Therefore Area of the quadrilateral OACB = 2 Area of the triangle OAC.

$OA = OB = \sqrt{S_1} = \sqrt{c}$ (Length of the tangent from the origin) and, CA = CB = $\sqrt{g^2+f^2-c}$ (Radius of the circle)

$\therefore$ Area of the quadrilateral OACB = 2 Area of the triangle OAC

$= 2 \times \left(\frac{1}{2}\right) OA \times CA \quad = \sqrt{c}\sqrt{g^2+f^2-c}$

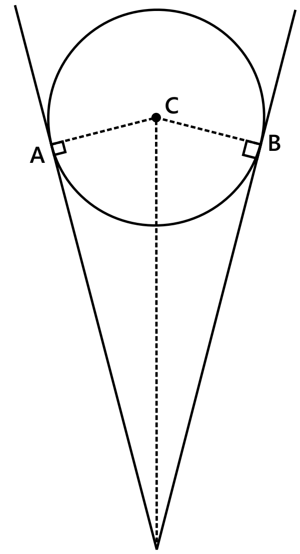


Figure 9.28

Illustration 16: The locus of a point which moves such that the tangents from it to the two circles $x^2 + y^2 - 5x - 3 = 0$ and $3x^2 + 3y^2 + 2x + 4y - 6 = 0$ are equal, is **(JEE MAIN)**

(A) $7x + 4y - 3 = 0$ (B) $17x + 4y + 3 = 0$ (C) $3x - 4y + 9 = 0$ (D) $13x - 4y + 15 = 0$

Sol: (B) Use the formula for length of tangent. Let P(h, k) be any point on the locus.

The length of the tangent from P to the first circle is $\sqrt{h^2+k^2-5h-3}$, ...(i)

Similarly, the length of the tangent to the other circle is $\sqrt{h^2+k^2+\frac{2}{3}h+\frac{4}{3}k-\frac{6}{3}}$. ...(ii)

On equating (i) and (ii), we get $17h + 4k + 3 = 0$,

Therefore, the required locus is $17x + 4y + 3 = 0$.

7.6 Pair of Tangents

From a given point $P(x_1, y_1)$ two tangents PA and PB can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$.

The combined equation of the pair of tangents is

$SS_1 = T^2$, where

S = 0 is the equation of circle,

T = 0 is the equation of tangent at (x_1, y_1), and

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

(S_1 is obtained by replacing x by x_1 and y by y_1 in S)

Pair of tangents from point (0, 0) to the circle are at right angles if $g^2 + f^2 = 2c$.

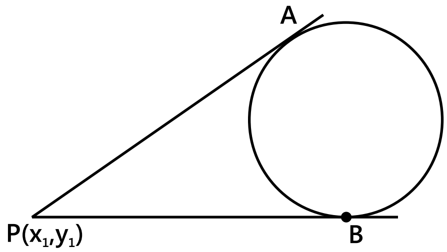


Figure 9.29

Illustration 17: Find the equation of the pair of the tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1). **(JEE MAIN)**

Sol: Here $(x_1, y_1) = (0, 1)$. So by using formula $SS_1 = T^2$ we can get required equation, where $S = x^2+y^2-2x+4y=0$, $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $T = xx_1 + yy_1 - (x+x_1) + 2(y+x_1)$.

Given circle is $S \equiv x^2+y^2-2x+4y=0$ ...(i)

Let P be the point (0, 1).

$\therefore S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \qquad \Rightarrow S_1 \equiv 0^2 + 1^2 - 2.0 + 4.1 = 5$

And, $T \equiv xx_1 + yy_1 - (x+x_1) + 2(y+x_1) \qquad \Rightarrow T \equiv x(0) + y(1) - (x + 0) + 2(y + 1)$

i.e., $T \equiv -x + 3y + 2$.

Hence, the equation of pair of tangents from P (0, 1) to the given circle is $SS_1 = T^2$

i.e. $5\left(x^2+y^2-2x+4y\right) = \left(-x+3y+2\right)^2$

$\Rightarrow 5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

$\Rightarrow 4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

$\Rightarrow 2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ...(ii)

Note: From (ii), we have $2x^2 + 3(y - 1)x - (2y^2 - 4y + 2) = 0$.

This is a quadratic equation in x, hence by using quadratic formula we get

$x = \dfrac{3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$ or, $4x - 3y + 3 = \pm\sqrt{25y^2 - 50y + 25}$

or, $4x - 3y + 3 = \pm 5(y - 1)$.

$\therefore$ Separate equations of tangents are $x - 2y + 2 = 0$ and $2x + y - 1 = 0$.

Illustration 18: From a point on the line $4x - 3y = 6$ tangents are drawn to the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ which make an angle of $\tan^{-1}\dfrac{24}{7}$ between them. Find the coordinates of all such points and the equations of tangents. **(JEE ADVANCED)**

Sol: Consider a point P on the line $4x - 3y = 6$ and use the formula.

Let $P(x_1, y_1)$ be a point on the line $4x - 3y = 6$.

If θ is the angle between the tangents, then $\tan\theta = \dfrac{24}{7}$.

For the given circle, Centre $C = (3, 2)$ and

Radius $= CA = \sqrt{3^2 + 2^2 - 4} = 3$ for $\tan\theta$

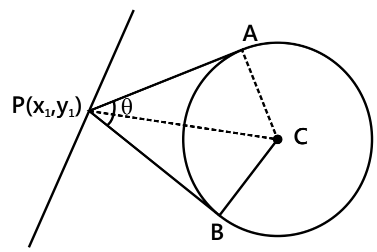


Figure 9.30

$\therefore$ The length of tangent, $PA = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4}$

$\therefore \tan\dfrac{\theta}{2} = \dfrac{AC}{PA} = \dfrac{3}{\sqrt{S_1}}$

$\Rightarrow \tan^2\dfrac{\theta}{2} = \dfrac{9}{S_1}$ or, $\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \dfrac{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4 - 9}{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4 + 9}$ or $\dfrac{1 - \frac{9}{-S_1}}{1 + \frac{9}{S_1}} = \dfrac{S_1 - 9}{S_1 + 9} = \dfrac{7}{25} \Rightarrow S_1 = 16$

$\therefore \dfrac{7}{25} = \dfrac{x_1^2 + y_1^2 - 6x_1 - 4y_1 - 5}{x_1^2 + y_1^2 - 6x_1 - 4y_1 + 13}$ $\left(\because \tan\theta = \dfrac{24}{7}\right)$

or, $x_1^2 + y_1^2 - 6x_1 - 4y_1 - 12 = 0$ ... (i)

As (x_1, y_1) is on the line $4x - 3y = 6$, we get $4x_1 - 3y_1 = 6$ ... (ii)

Solving (i) and (ii), we get

$x_1^2 + \left(\dfrac{4x_1 - 6}{3}\right)^2 - 6x_1 - 4\left(\dfrac{4x_1 - 6}{3}\right) - 12 = 0$

$\Rightarrow 9x_1^2 + (4x_1 - 6)^2 - 54x_1 - 12(4x_1 - 6) - 108 = 0$

$\Rightarrow 25x_1^2 - 150x_1 = 0$ $\Rightarrow x_1(x_1 - 6) = 0$

$\Rightarrow x_1 = 0, 6$ and, $y_1 = \dfrac{4x_1 - 6}{3} = -\dfrac{6}{3}, \dfrac{18}{3} = -2, 6.$

$\therefore (x_1, y_1) \equiv (0, -2)$ and $(6, 6)$.

The equation of the pair of tangents is given by $SS_1 = T^2$

where $S \equiv x^2 + y^2 - 6x - 4y + 4$,

$S_1 = x_1^2 + y_1^2 - 6x_1 - 4y_1 + 4$, and

$T = xx_1 + yy_1 - 3(x + x_1) - 2(y + y_1) + 4$

$\therefore$ The pair of tangents from (0, –2) is

$(x^2 + y^2 - 6x - 4y + 4)\cdot(0 + 4 - 0 + 8 + 4) = (0 + y(-2) - 3(x) - 2(y - 2) + 4)^2$

$\Rightarrow 16(x^2 + y^2 - 6x - 4y + 4) = (-3x - 4y + 8)^2$

$\Rightarrow 16(x^2 + y^2 - 6x - 4y + 4) = 9x^2 + 16y^2 + 64 + 24xy - 48x - 64y$

$\Rightarrow 7x^2 - 24xy - 48x = 0$ $\quad\quad\Rightarrow x(7x - 24y - 48) = 0$

$\therefore$ The tangents from (0, –2) are x = 0, and 7x –24y –48 =0.

Similarly, the equation of the pair of tangents from (6, 6) is

$(x^2 + y^2 - 6x - 4y + 4)\cdot(36 + 36 - 6\cdot 6 - 4\cdot 6 + 4) = \{x\cdot 6 + y\cdot 6 - 3(x + 6) - 2(y + 6) + 4\}^2$

$\Rightarrow 16(x^2 + y^2 - 6x - 4y + 4) = (3x + 4y - 26)^2 = 9x^2 + 16y^2 + 676 + 24xy - 156x - 208y$

$\Rightarrow 7x^2 - 24xy + 60x + 144y - 612 = 0$

$\Rightarrow (7x - 24y + 102)(x - 6) = 0$

$\therefore$ The tangents from (6, 6)are x – 6 = 0, and 7x – 24y + 102 =0.

Illustration 19: Obtain the locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ which include an angle α. **(JEE ADVANCED)**

Sol: Consider (x_1, y_1) as the point of intersection of tangents to the given circle and then use $\tan\frac{\alpha}{2} = \frac{a}{\sqrt{S_1}}$ to get the desired result.

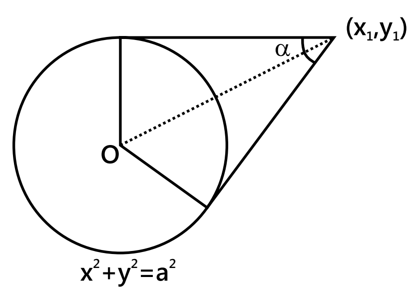


Figure 9.31

Let (x_1, y_1) be the point of intersection of a pair of tangents to the given circle.

If the pair of straight lines includes an angle α, then

$\Rightarrow \tan\frac{\alpha}{2} = \frac{a}{\sqrt{S_1}}$ $\quad\quad\Rightarrow \tan\alpha = \frac{2\frac{a}{\sqrt{S_1}}}{1 - \frac{a^2}{S_1}}$

$\Rightarrow \tan\alpha = \frac{2a\sqrt{x_1^2 + y_1^2 - a^2}}{y_1^2 + x_1^2 - 2a^2}$

$\Rightarrow \left(x_1^2 + y_1^2 - 2a^2\right)^2 \tan^2\alpha = 4a^2\left(x_1^2 + y_1^2 - a^2\right)$

Hence, the required locus is $(x^2 + y^2 - 2a^2)^2 \tan^2\alpha = 4a^2(x^2 + y^2 - a^2)$.

7.7 Director Circle

The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle.

For the circle $x^2 + y^2 = a^2$, the equation of the director circle is $x^2 + y^2 = 2a^2$.

Hence, the centre of the director circle is same as the centre of the given circle, and the radius is $\sqrt{2}$ times the radius of the given circle.

General Form: For the general form of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the equation of the director circle is given by $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

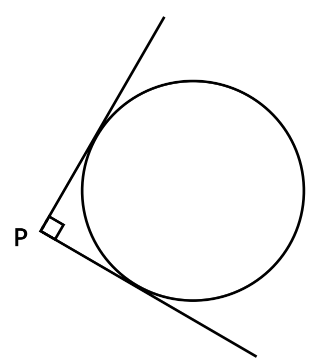


Figure 9.32

Illustration 20: Find the equation of the director circle of the circle $(x-2)^2 + (y+1)^2 = 2$. **(JEE MAIN)**

Sol: As we know, for the circle $x^2 + y^2 = a^2$, the equation of the director circle is $x^2 + y^2 = 2a^2$.

For the given circle, Centre $\equiv(2, -1)$ & Radius = $\sqrt{2}$.

$\therefore$ The centre of the director circle $\equiv$ (2, – 1), and the radius of the director circle = $\sqrt{2}\times\sqrt{2} = 2$.

$\therefore$ The required equation is $(x-2)^2 + (y+1)^2 = 4$.

8. NORMALS

The normal of a circle at any point is a straight line, perpendicular to the tangent and passing through the centre of the circle.

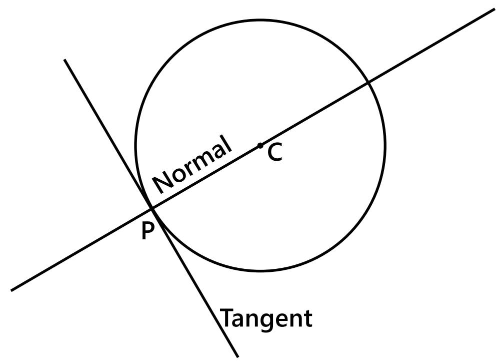


Figure 9.33

(a) Equation of normal: The equation of normal to the general form of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point

(x_1, y_1) on the circle is

$(y - y_1) = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$ or, $\dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$

The equation of normal to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is $xy_1 - xy_1 = 0$ or, $\dfrac{x}{x_1} = \dfrac{y}{y_1}$.

(b) Parametric Form: Equation of normal at $(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = a^2$ is $\dfrac{x}{a\cos\theta} = \dfrac{y}{a\sin\theta}$

or, $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta}$ or, $y = x\tan\theta$ or, $y = mx$ (where $m = \tan\theta$).

Illustration 21: Find the equation of the circle having the pair of lines $x^2 + 2xy + 3x + 6y = 0$ as its normal and having the size just sufficient to contain the circle $x(x-4) + y(y-3) = 0$. **(JEE ADVANCED)**

Sol: By solving equation $x^2 + 2xy + 3x + 6y = 0$ we will get point of intersection of normals i.e. centre of required circle. As given circle $x(x-4) + y(y-3) = 0$ lies inside the required circle hence distance between centres will be equal to the difference between their radius, therefore we can find out radius of required circle by using distance formula.

Given the equation of pair of normal is $x(x+3) + 2y(x+3) = 0$

$\Rightarrow (x+3)(x+2y) = 0$

$\therefore$ Either $(x+3)=0$ ...(i) or $(x+2y)=0$ ...(ii)

On solving (i) and (ii), we get $x = -3$ and $y = \dfrac{3}{2}$

$\therefore$ The centre $\equiv \left(-3, \dfrac{3}{2}\right)$ (The point of intersection of the normals).

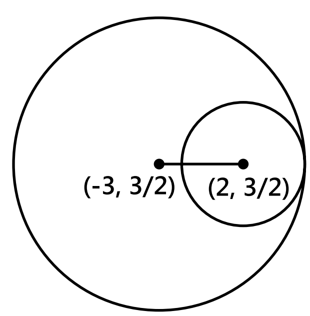


Figure 9.34

For the circle $x^2 + y^2 - 4x - 3y = 0$ ...(iii)

centre = $\left(2, \dfrac{3}{2}\right)$ and radius, $r = \sqrt{(-2)^2 + \left(\dfrac{-3}{2}\right)^2 - 0} = \dfrac{5}{2}$.

If the circle $x^2 + y^2 - 4x - 3y = 0$ lies inside another circle of radius 'a', then

$a - r$ = distance between the centres $\left(-3, \dfrac{3}{2}\right)$ and $\left(2, \dfrac{3}{2}\right)$

$\Rightarrow a - \dfrac{5}{2} = \sqrt{(-3-2)^2 + \left(\dfrac{3}{2} - \dfrac{3}{2}\right)^2}$ $\Rightarrow a = 5 + \dfrac{5}{2}$ $\therefore a = \dfrac{15}{2}$.

Hence, the equation of the circle is $(x + 3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2$ or, $x^2 + y^2 + 6x - 3y = 45$.

9. CHORD OF CONTACT

Consider a point $P(x_1, y_1)$ lying outside the circle. Tangents are drawn to touch the given circle at Q and R respectively (as shown in the diagram). The chord joining the points of contact of the two tangents to a circle (or any conic) from the point P, outside it, is known as the chord of contact.

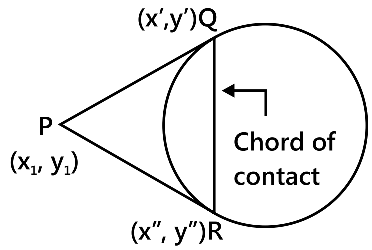


Figure 9.35

9.1 Equation of Chord of contact

The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$. Equation of chord of contact at (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Clearly, the equation of the chord of contact coincides with the equation of the tangent.

Length of chord of contact

Consider a circle of radius 'r' and the length of perpendicular from the centre to the chord of contact be 'p', then the length of the chord, QR = $2\sqrt{r^2 - p^2}$.

PLANCESS CONCEPTS

- Area of $\Delta PQR = \frac{1}{2} \times PM \times QR = \frac{a\left(x_1^2 + y_1^2 - a^2\right)^{3/2}}{x_1^2 + y_1^2} = \frac{RL^3}{R^2 + L^2}$

  where, the length of the tangent, L = $\sqrt{x_1^2 + y_1^2 - a^2}$ and, radius of circle, R = a.

- Equation of circle circumscribing the triangle PQR is $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

  Note: Circumscribing Circle also passes through centre of original Circle

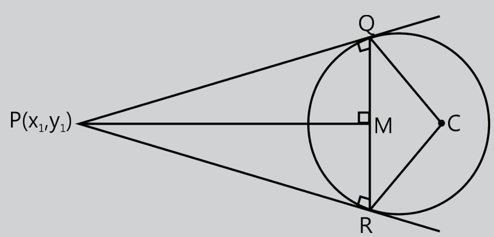


Figure 9.36

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 22: Find the equation of the chord of contact of the tangents drawn from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$. **(JEE MAIN)**

Sol: Equation of chord of contact is T = 0

Given circle is $S \equiv x^2 + y^2 - 2x + 4y + 7 = 0$ ...(i)

For point $P \equiv (1, 2)$,

$S1 > 0, \Rightarrow$ the point P lies outside the circle. and, $T \equiv x(1) + y(2) - (x + 1) + 2(y + 2) + 7$ i.e. $T \equiv 4y + 10$

$\therefore$ The equation of the chord of contact is T = 0 i.e. $2y + 5 = 0$.

Illustration 23: The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2 + y^2 = a^2$ which touches the circle $x^2 + y^2 - 2ax = 0$ passes through the point **(JEE ADVANCED)**

(A) $\left(\frac{a}{2}, 0\right)$ (B) $\left(0, \frac{a}{2}\right)$ (C) (0, a) (D) (a, 0)

Sol: (A) and (C) Apply the condition of tangency to the equation of chord of contact.

Let P (h, k) be the point of intersection of the tangents at the extremities of the chord AB of the circle $x^2 + y^2 = a^2$.

$\therefore$ The equation of the chord of contact AB w.r.t. the point P is $hx + ky = a^2$.

The line $hx + ky = a^2$ touches the circle $x^2 + y^2 - 2ax = 0$ if $\left|\frac{h(a)+k(0)-a^2}{\sqrt{h^2+k^2}}\right| = a$

$\Rightarrow (h - a)^2 = h^2 + k^2$

Therefore, the locus of (h, k) is $(x - a)^2 = x^2 + y^2$ or, $y^2 = a(a - 2x)$.

Clearly, points (A) and (C) satisfy the above equation.

9.2 Chord Bisected at a given Point

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S_1$.

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

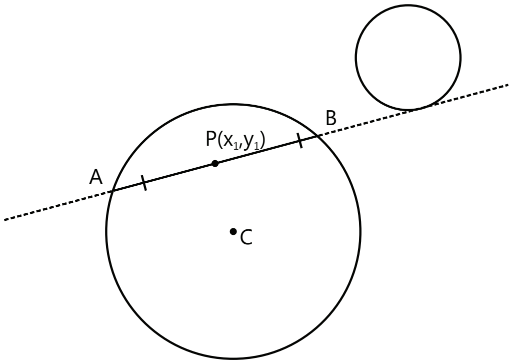


Figure 9.37: Chord bisected by point P

PLANCESS CONCEPTS

The smallest chord of a circle passing through a point 'M' at a maximum distance from the centre is the one whose middle point is M.

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 24: Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose middle point is (1, –1). **(JEE MAIN)**

Sol: Use $T = S_1$

Given, $S \equiv x^2 + y^2 + 6x + 8y - 11 = 0$

For point L (1, 1), $S_1 = 1^2 + (-1)^2 + 6.1 + 8(-1) - 11 = -11$ and

$T = x.1 + y.(-1) + 3(x + 1) + 4(y - 1) - 11$ i.e. $T = 4x + 3y - 12$

Now equation of the chord having middle point, L(1, –1) is

$\therefore$ $4x + 3y - 12 = -11$ $\Rightarrow 4x + 3y - 1 = 0$

Second method:

Let C be the centre of the given circle, $C \equiv (-3, -4)$

$\therefore$ Slope of CL = $\frac{-4+1}{-3-1} = \frac{3}{4}$

$\therefore$ Equation of chord whose middle point is L, is

$\therefore \quad y + 1 = -\frac{4}{3}(x - 1) \qquad [\because$ chord is perpendicular to CL]

Or, $\quad 4x + 3y - 1 = 0$

Illustration 25: Find the locus of the middle points of chords of given circle $x^2 + y^2 = a^2$ which subtends a right angle at the fixed point (p, q). **(JEE ADVANCED)**

Sol: As M(h, k)be the midpoint of the chord AB which subtends an angle of 90^0 at the point N (p , q) therefore a circle can be drawn with AB as the diameter and passing through the point N, hence AM = MN.

$\therefore$ AM = MN $\qquad \Rightarrow AM^2 = MN^2 \qquad \Rightarrow a^2 - (h^2 + k^2) = (h - p)^2 + (k - q)^2$

$\Rightarrow a^2 - h^2 - k^2 = h^2 + p^2 - 2hp + k^2 + q^2 - 2kq$

$\Rightarrow 2h^2 + 2k^2 - 2ph - 2qk + p^2 + q^2 - a^2 = 0$

$\Rightarrow h^2 + k^2 - ph - qk + \frac{1}{2}\left(p^2 + q^2 - a^2\right) = 0$

Hence, the required locus is $x^2 + y^2 - px - qy + \frac{1}{2}\left(p^2 + q^2 - a^2\right) = 0$.

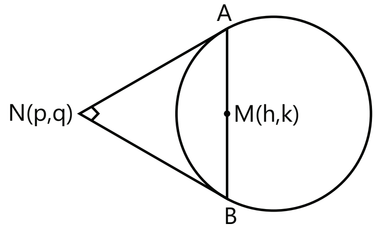


Figure 9.38

10. COMMON CHORD OF TWO CIRCLES

Definition: The chord joining the points of intersection of two given circles is called their common chord.

Equation of common chord: The equation of the common chord of two circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ .... (i)

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ .... (ii)

is given by $S_1 - S_2 = 0$ i.e., $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$.

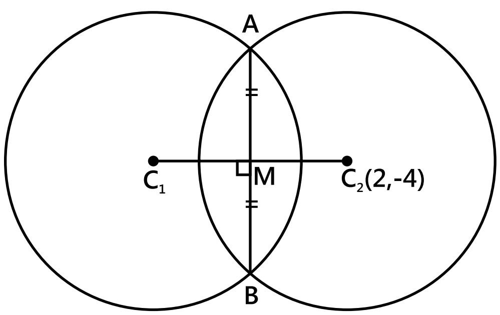


Figure 9.39: Common Chord

Length of the common chord: AB = 2(AM) = $2\sqrt{C_1A^2 - C_1M^2}$

Where, C_1A = radius of the circle $S_1 = 0$, and

C_1M = length of the perpendicular from the centre C_1 to the common chord AB.

Note: If the two circles touch each other, then the length of common chord is zero and the common chord is the common tangent to the two circles at the point of contact.

PLANCESS CONCEPTS

The length of the common chord AB is maximum when it is the diameter of the smallest circle.

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 26: Find the equation and the length of the common chord of two circles.

$2x^2 + 2y^2 + 7x - 5y + 2 = 0$ and $x^2 + y^2 - 4x + 8y - 18 = 0$ **(JEE MAIN)**

Sol: Use the formula for equation of common chord and length of common chord. Equation of common chord of circle is $S_1 - S_2 = 0$ i.e., $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ and length of common chord is $2\sqrt{C_2A^2 - C_2M^2}$.

Given $S_1 = x^2 + y^2 + \frac{7}{2}x - \frac{5}{2}y + 1 = 0$ .... (i)

$S_2 = x^2 + y^2 - 4x + 8y - 18 = 0$ .... (ii)

Therefore, the equation of the common chord AB is $S_1 - S_2 = 0$

i.e. $\frac{15}{2}x - \frac{21}{2}y + 19 = 0 \Rightarrow 15x - 21y + 38 = 0$ .... (iii)

The length of the perpendicular from the centre $C_2(2, -4)$ to the common chord AB is $C_2M = \left|\frac{30+84+38}{\sqrt{15^2+21^2}}\right| = \frac{152}{\sqrt{666}}$

Radius of the circle $S_2 = 0$ is, $C_2A = \sqrt{38}$

$\therefore$ The length of the common chord = AB = 2AM

$= 2\sqrt{C_2A^2 - C_2M^2}$; $= 2\sqrt{38 - \left(\frac{152}{\sqrt{666}}\right)^2} = 2\sqrt{\frac{1102}{333}}$

Illustration 27: Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$; find the point of intersection of these tangents. **(JEE ADVANCED)**

Sol: As we know that, if (x_1, y_1) is a point of intersection of tangents of circle $x^2 + y^2 = a^2$ then equation of chord of contact is $xx_1 + yy_1 = a^2$ and the equation of common chord of two circles are $S_1 - S_2 = 0$ i.e., $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$. By using these two formulae we can solve the problem.

Given circles are $S_1 \equiv x^2 + y^2 - 12 = 0$ ... (i)

and $S_2 \equiv x^2 + y^2 - 5x + 3y - 2 = 0$ ... (ii)

The equation of common chord is $S_1 - S_2 = 0$ i.e. $5x - 3y - 10 = 0$ ... (iii)

Let this line meet circle (i) at A and B, and $P(\alpha, \beta)$ be the point of intersection of the tangents at A and B. Therefore, the equation of the chord of contact AB is $x\alpha + y\beta - 12 = 0$ ... (iv)

As (iii) and (iv) represent the same line, therefore on comparison, we get

$\frac{\alpha}{5} = \frac{\beta}{-3} = \frac{6}{5}$ $\therefore$ $\alpha = 6$ and $\beta = -\frac{18}{5}$. Hence, $P \equiv \left(6, -\frac{18}{5}\right)$.

11. DIAMETER OF CIRCLE

The locus of the middle points of a system of parallel chords of a circle is known as the diameter of the circle.

Let the equation of parallel chords be

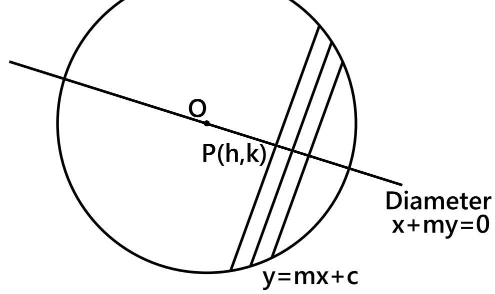


Figure 9.40

$y = mx + c$ (where, c is a parameter).

$\therefore$The equation of the diameter bisecting parallel chords

of the circle $x^2 + y^2 = a^2$ is given by $x + my = 0$.

12. POLE AND POLAR

Let P (x_1,y_1) be any point inside or outside the circle. Passing through the point P chords AB and A′ B′ are drawn. If the tangents at point A and point B intersect at Q (h, k) , then the locus of Q is a straight line and is called the polar of point P with respect to circle and P is called the pole. Similarly, if the tangents to the circle at A′ and B′ meet at Q′, then the locus of Q′ is the polar with P as its pole.

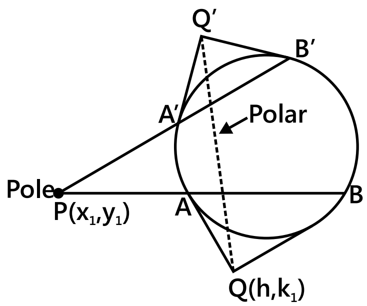


Figure 9.41(a): Polar of a point P outside the circle

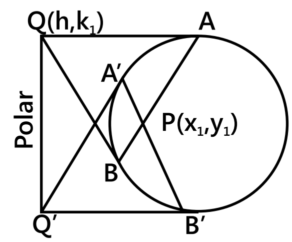


Figure 9.41(b): Polar of a point P inside the circle

Equation of polar of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ w.r.t. point P (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ i.e. T = 0.

If the circle is $x^2 + y^2 = a^2$, then its polar w.r.t. (x_1, y_1) is $xx_1 + yy_1 – a^2 = 0$ i.e. T = 0.

Pole of a line w.r.t. the circle $x^2 + y^2 = a^2$

Consider a line $lx + my + n = 0$ and let (x_1, y_1) be the pole of the line w.r.t. the circle $x^2 + y^2 = a^2$.

For the point (x_1, y_1),

The equation of polar w.r.t. the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 – a^2 = 0$.

Since $lx + my + n = 0$ and $xx_1 + yy_1 – a^2 = 0$ represent the same line.

$$\therefore \frac{x_1}{l} = \frac{y_1}{m} = \frac{-a^2}{n} \quad \Rightarrow x_1 = -\frac{a^2 l}{n} \text{ and } y_1 = -\frac{a^2 m}{n}.$$

Hence, the pole of the line $lx + my + n = 0$ is $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$

Pole of a line w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Consider a line $lx + my + n = 0$.

If (x_1, y_1) is the pole, then the equation of polar is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Now, since $lx + my + n = 0$ and $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ represent the same line,

$$\therefore \frac{x_1+g}{l} = \frac{y_1+f}{m} = \frac{gx_1+fy_1+c}{n}$$

On simplification, we get $\frac{x_1+g}{l} = \frac{y_1+f}{m} = \frac{g^2+f^2-c}{lg+fm-n}$

$$\Rightarrow \frac{x_1+g}{l} = \frac{y_1+f}{m} = \frac{r^2}{lg+mf-n},$$ where r is radius of the circle.

12.1 Conjugate Points and Conjugate Lines

(a) If the polar of point P (x_1, y_1) w.r.t. a circle $x^2 + y^2 = a^2$, passes through $Q(x_2, y_2)$, then the polar of Q will pass through P. Such points are called conjugate points and they satisfy the relation $x_1x_2 + y_1y_2 = a^2$

(b) If the pole of the line $l_1x + m_1y + n_1 = 0$ w.r.t. a circle lies on another line $l_2x + m_2y + n_2 = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

Consider the circle $x^2 + y^2 = a^2$,

The pole P of the line $l_1x + m_1y + n_1 = 0$ w.r.t. the circle is given by $\left(-\frac{a^2 l_1}{n_1}, -\frac{a^2 l_1}{n_1}\right)$.

$\Rightarrow l_2\left(-\frac{a^2 l_1}{n_1}\right) + m_2\left(-\frac{a^2 m_1}{n_1}\right) + n_2 = 0 \qquad \therefore a^2\left(l_1 l_2 + m_1 m_2\right) = n_1 n_2$

PLANCESS CONCEPTS

Points P (x_1, y_1) and Q (x_2, y_2) are conjugate points w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ if $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

If P and Q are conjugate points w.r.t. a circle with centre at O and radius 'a' then $PQ^2 = OP^2 + OQ^2 - 2a^2$.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 28: Find the pole of the line $3x + 5y + 17 = 0$ with respect to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$. **(JEE MAIN)**

Sol: If $P(\alpha, \beta)$ be the pole of line with respect to the given circle. Then the equation of polar of point $P(\alpha, \beta)$ w.r.t. the circle is $x\alpha + y\beta + 2(x + \alpha) + 3(y + \beta) + 9 = 0$. And this equation represent same line which is represented by equation $3x + 5y + 17 = 0$. By solving these two equation simultaneously we will get required pole.

Given circle is $x^2 + y^2 + 4x + 6y + 9 = 0$ ... (i)

and, given line is $3x + 5y + 17 = 0$ ... (ii

$\Rightarrow (\alpha + 2)x + (\beta + 3)y + 2\alpha + 3\beta + 9 = 0$ ... (iii)

Since equation (ii) and (iii) represent the same line,

$\therefore \frac{\alpha+2}{3} = \frac{\beta+3}{5} = \frac{2\alpha+3\beta+9}{17} \Rightarrow 5\alpha + 10 = 3\beta + 9$

$\Rightarrow 5\alpha - 3\beta = -1$ ... (iv)

and, $17\alpha + 34 = 6\alpha + 9\beta + 27 \qquad \Rightarrow 11\alpha - 9\beta = -7$ ... (v)

From (iv) and (v), we get $\alpha = 1, \beta = 2$

Hence, the pole of the line $3x + 5y + 17 = 0$ w.r.t. the circle $x^2 + y^2 + 4x + 6y + 9 = 0$ is (1, 2).

Illustration 29: A variable circle is drawn to touch the axis of x at origin. Find locus of pole of straight line $lx + my + n = 0$ w.r.t. circle. **(JEE ADVANCED)**

Sol: As circle touches x-axis at origin therefore let $(0, \lambda)$ be its centre then equation of circle will be $x^2 + (y - \lambda)^2 = \lambda^2$. Hence by considering $P(x_1, y_1)$ be the pole and using polar equation we will get required result.

Let the centre of the circle be $(0, \lambda)$.

Then the equation of the circle is $x^2 + (y - \lambda)^2 = \lambda^2$

$\Rightarrow x^2 + y^2 - 2\lambda y = 0.$

Let $P(x_1, y_1)$ be the pole of the line $lx + my + n = 0$ w.r.t. the circle,

then, the equation of the polar is $xx_1 + yy_1 - \lambda(y + y_1) = 0$

$xx_1 + y(-\lambda + y_1) - \lambda y_1 = 0$

$\therefore$ On comparison, we get $\frac{x_1}{l} = \frac{-\lambda + y_1}{m} = \frac{-\lambda y_1}{n}$.

Hence, the locus of the pole is $ly^2 = mxy - xn$.

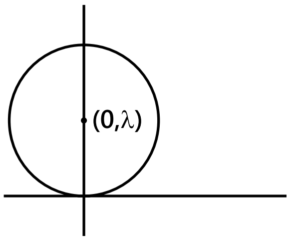


Figure 9.42

Illustration 30: Prove that if two lines at right angles are conjugate w.r.t. circle then one of them passes through centre. **(JEE ADVANCED)**

Sol: Let two perpendicular lines which are conjugate to each other be

$ax + by + c = 0$ ... (i)

$bx - ay + \lambda = 0$ ... (ii)

$\therefore$ The equation of the polar of a point (x_1, y_1) is $xx_1 + yy_1 - r^2 = 0$ ... (iii)

On comparing (i) and (iii), we get $\frac{x_1}{a} = \frac{y_1}{b} = \frac{-r^2}{c}$.

From the definition of conjugate lines, we know that the point (x_1, y_1) should satisfy the equation $bx - ay + \lambda = 0$,

hence $\frac{-br^2a}{c} + \frac{ar^2b}{c} + \lambda = 0 \quad \Rightarrow \lambda = 0.$

Therefore, $bx - ay + \lambda = 0$ passes through (0, 0).

13. COMMON TANGENTS TO TWO CIRCLES

Different cases of intersection of two circles:

Let the two circles be $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ ... (i) and, $(x - x_1)^2 + (y - y_2)^2 = r_2^2$ ... (ii)

Then following cases may arise:

Case I: When the distance between the centres is greater than the sum of radii. $\boxed{C_1C_2 > r_1 + r_2}$

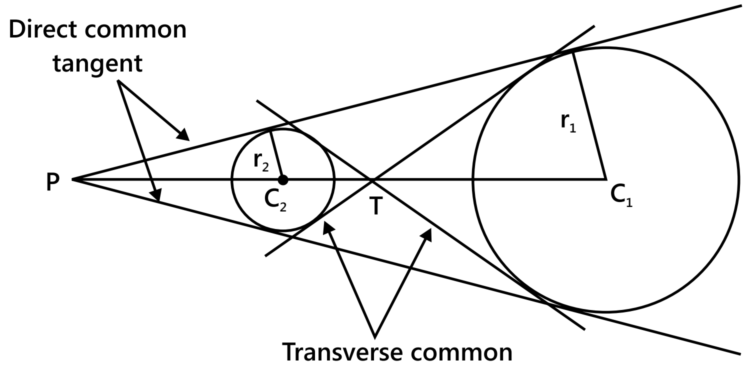


Figure 9.43: Common tangents for non-intersecting and non-overlapping circles

In this case four common tangents can be drawn, in which two are direct common tangents and the other two are transverse common tangents.

The points P and T , the point of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles. The point P and T divide the join of C_1 and C_2 externally and internally respectively in the ratio $r_1 : r_2$.

i.e. $\frac{C_1P}{C_2P} = \frac{r_1}{r_2}$ (externally) and $\frac{C_1T}{C_2T} = \frac{r_1}{r_2}$ (internally)

$\therefore P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$ and $T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$.

Steps to find Equations of Common Tangents

Let the equation of tangent of any circle in the slope form be (y+f) = m(x+g) + $a\sqrt{1+m^2}$ where, a is radius of circle and m is the slope of tangent.

The value of 'm' can be obtained by substituting the co-ordinates of the point P and T in the above equation.

Note: Length of an external (or direct) common tangent, $L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$, and Length of an internal (or transverse) common tangent, $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$. where, d is the distance between the centres of the two circles, and r_1, r_2 are the radii of the two circles. Therefore, the length of internal common tangent is always less than the length of the external common tangent.

Case-II: When the distance between the centres is equal to the sum of radii (Circles touching externally)

$\boxed{C_1 C_2 = r_1 + r_2}$

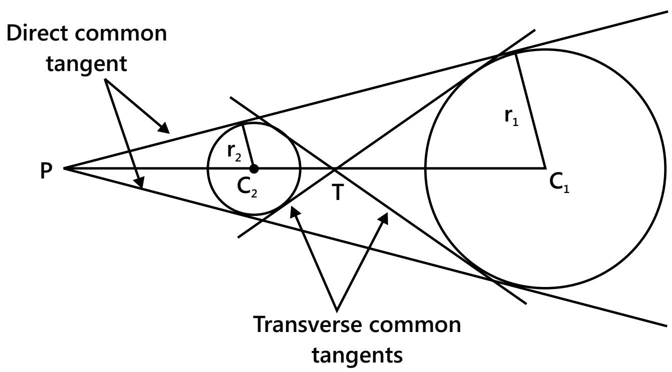


Figure 9.44: Common tangents of circles touching externally

In this case three common tangents can be drawn, two direct common tangents and one transverse common tangent.

Case III: When the distance between the centres is less than the sum of radii. (Intersecting circles)

$\boxed{|r_1 - r_2| < C_1 C_2 < r_1 + r_2}$

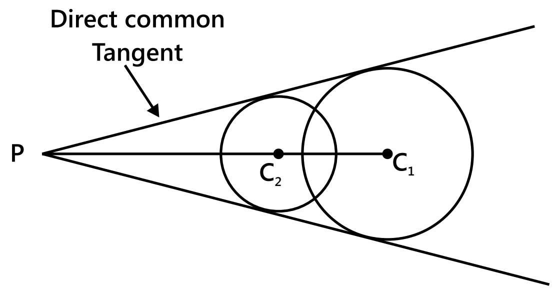


Figure 9.45: Common tangents for intersecting circles

In this case two direct common tangents can be drawn as shown in the diagram.

Case IV: When the distance between the centres is equal to the difference of the radii. (Circles touching each other internally), i.e. $|C_1C_2| = |r_1 - r_2|$.

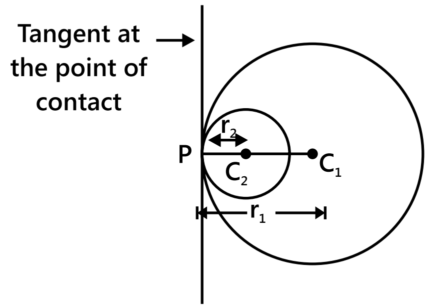


Figure 9.46: Common tangents for circles touching each other internally

In this case the total number of common tangents is one.

Case V: When the distance between the centres is less than the difference of the radii. (Circles neither touch each other nor intersect), i.e. $|C_1C_2| < |r_1 - r_2|$.

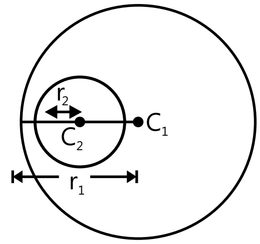


Figure 9.47

In this case, the number of common tangents is zero.

Illustration 31: Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally. **(JEE MAIN)**

Sol: When distance between centre of circle is equal to the sum of their radius then they touches eachother externally and when it is equal to the difference of their radius then circle touches eachother internally.

Let C_1 and C_2 be the centres and r_1 and r_2 the radii of $S_1 \equiv x^2 + y^2 - 2x - 4y = 0$ and $S_1 \equiv x^2 + y^2 - 8y - 4 = 0$ respectively.

$\therefore C_1 \equiv (1, 2), C_2 \equiv (0, 4), r_1 = \sqrt{5}, r_2 = 2\sqrt{5}$

Now, $\quad C_1C_2 = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$ and

$\quad r_1 + r_2 = 3\sqrt{5}, |r_1 - r_2| = \sqrt{5}$

Thus, $C_1C_2 = |r_1 - r_2|$, hence the two circles touch each other internally,

Illustration 32: Prove that the two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other,

if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$ **(JEE ADVANCED)**

Sol: Two circles touch each other if distance between centres of these two circles are equal to the sum or difference of their radius.

Let centres of given circles be C_1 and C_2 and their radii be r_1 and r_2 respectively.

$\therefore C_1 \equiv (-a, 0);\ r_1 = \sqrt{a^2 - c}$ and $C_2 \equiv (0, -b);\ r_2 = \sqrt{b^2 - c}$

Two circles touch each other, if $C_1C_2 = r_1 \pm r_2$

$\Rightarrow \sqrt{a^2 + b^2} = \sqrt{a^2 - c} \pm \sqrt{b^2 - c} \Rightarrow a^2 + b^2 = a^2 - c + b^2 - c \pm 2\sqrt{(a^2 - c)(b^2 - c)}$

$\Rightarrow c = \pm\sqrt{a^2b^2 - a^2c - b^2c + c^2} \Rightarrow c^2 = a^2b^2 - a^2c - b^2c + c^2$

$\Rightarrow a^2c + b^2c = a^2b^2$ $\therefore \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$.

Illustration 33: An equation of a common tangent to the circles $x^2 + y^2 + 14x - 4y + 28 = 0$ and $x^2 + y^2 - 14x + 4y - 28 = 0$ is **(JEE ADVANCED)**

(A) $x - 7 = 0$ (B) $y + 7 = 0$ (C) $28x + 45y + 371 = 0$ (D) None of these

Sol: (C) Calculate the distance between the centres and use the different cases of two circles.

Let $S_1 \equiv x^2 + y^2 + 14x - 4y + 28 = 0$ ...(i)

$\Rightarrow C_1 = (-7, 2)$ and $r_1 = 5$

and, $S_2 \equiv x^2 + y^2 - 14x + 4y - 28 = 0$ ...(ii)

$\Rightarrow C_2 = (7, -2)$ and $r_2 = 9$

$\therefore C_1C_2 = \sqrt{(7+7)^2 + (-2-2)^2} > r_1 + r_2$.

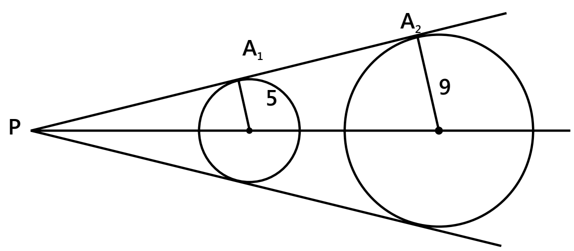


Figure 9.48

Hence, four common tangents are possible.

For $x - 7 = 0$, ... (iii)

Clearly, C_2 lies on the (iii).

For $y + 7 = 0$ ... (iv)

Length of perpendicular from $C_1 = 9 > r_1$.

For $28x + 45y + 371 = 0$ ... (v)

Length of perpendicular from $C_1 = \left|\frac{28(-7) + 45(2) + 371}{\sqrt{28^2 + 45^2}}\right| = \frac{265}{53} = r_1$.

Length of perpendicular from $C_2 = \left|\frac{28(7) + 45(-2) + 371}{\sqrt{28^2 + 45^2}}\right| = \frac{477}{53} = r_2$.

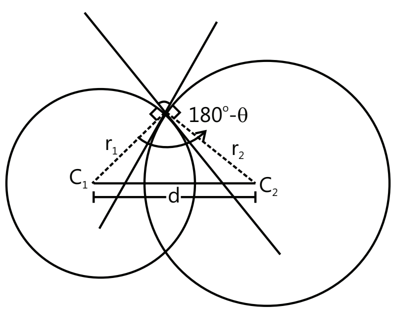


Figure 9.49

14. ANGLE OF INTERSECTION OF TWO CIRCLES

The angle of intersection between two circles S = 0 and S′ = 0 is defined as the angle between their tangents at their point of intersection.

If $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$;

$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by $\cos(180 - \theta) = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

or, $\cos(180 - \theta) = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_1}}$.

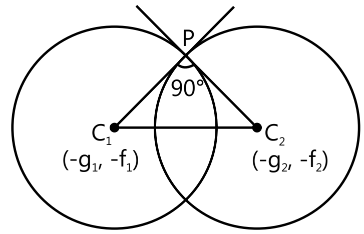


Figure 9.50: Angle of intersection

Condition of Orthogonality: Two circles are said to be orthogonal to each other if the angle of intersection of the two circles is 90°.

$\Rightarrow 2(g_1g_2 + f_1f_2) = c_1 + c_2$.

PLANCESS CONCEPTS

If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes though the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polar w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent in a circle which is orthogonal all the three circles.

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Illustration 34: If a circle passes through the point (3, 4) and cuts the circle $x^2 + y^2 = a^2$ orthogonally, the equation of the locus of its centre is **(JEE MAIN)**

(A) $3x + 4y - a^2 = 0$ (B) $6x + 8y = a^2 + 25$

(C) $6x + 8y + a^2 + 25 = 0$ (D) $3x + 4y = a^2 + 25$

Sol : (B)

As we know Two circle are said to be orthogonal if $2(g_1g_2 + f_1f_2) = c_1+c_2$. So by considering required equation of circle as $x^2 + y^2 + 2gx + 2fy + c = 0$ and As point (3, 4) satisfies this equation so by solving these two equation we will get required equation of the locus of its centre.

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

As the point (3, 4) lies on (i), we have $9 + 16 + 6g + 8f + c = 0$

$\Rightarrow 6g + 8f + c = -25$ ... (ii)

$\Rightarrow 2g\times0 + 2f\times0 = c - a^2$ $\Rightarrow c = a^2$.

$\therefore$ From equation (ii), we have $6g + 8f + a^2 + 25 = 0$.

Hence locus of the centre $(-g, -f)$ is $6x + 8y - (a^2 + 25) = 0$.

Illustration 35: Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 56 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4x + 1 = 0$. **(JEE ADVANCED)**

Sol: By considering the required circle to be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and using orthogonality formula $2(g_1g_2 + f_1f_2) = c_1+c_2$ we will get a relation between g and f. Also as the centre lies on the line $3x + 4x + 1 = 0$, by solving these equation we will get required result.

Let the required circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

Given $S_1 \equiv x^2 + y^2 + 3x - 5y + 56 = 0$ ... (ii)

and, $S_2 \equiv x^2 + y^2 - 7x + \frac{29}{4} = 0$. ... (iii)

Since (i) is orthogonal to (ii) and (iii)

$\therefore\ 2g\left(\frac{3}{2}\right) + 2f\left(-\frac{5}{2}\right) = c + 6\ + 2f$ $\Rightarrow 3g - 5f = c + 6$ ... (iv) and $2g\left(-\frac{7}{2}\right) + 2f.0 \Rightarrow c + \frac{29}{4}$ $\quad\Rightarrow -7g = c + \frac{29}{4}$ ...(v)

From (iv) and (v), we get $40g - 20f = -5$ ... (vi)

Given line is $3x + 4y = -1$ ... (vii)

$(-g, -f)$ also lies on the line (vii). $\Rightarrow -3g - 4f = -1$ ... (viii)

$\therefore$ $g = 0$, $f = \frac{1}{4}$ and $c = -\frac{29}{4}$ [From (vi) and (viii)]

$\therefore$ The equation of the circle is $x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0$ or, $4(x^2 + y^2) + 2y - 29 = 0$

15. FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the point of intersection of two given circle S = 0 and S′ = 0 is given by $S + \lambda S' = 0$, (where λ is a parameter, $\lambda \neq -1$)

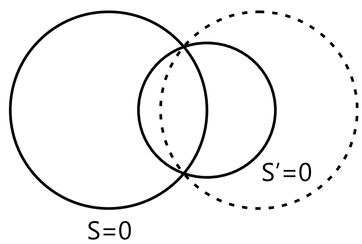


Figure 9.51

(b) The equation of the family of circles passing through the point of intersection of circle S = 0 and a line L = 0 is given by $S + \lambda L = 0$, (where λ is a parameter)

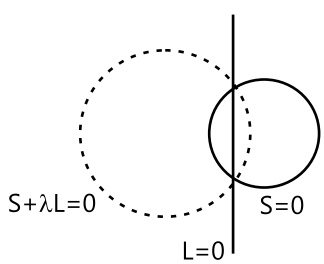


Figure 9.52

(c) The equation of the family of circles touching the circle S = 0 and the line L = 0 at their point of contact P is $S + \lambda L = 0$, (where λ is a parameter)

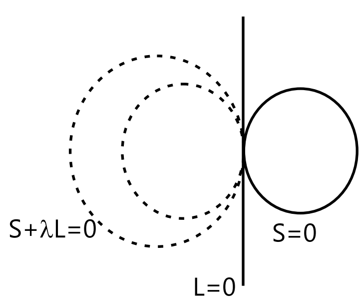


Figure 9.53

(d) The equation of a family of circles passing through two given points P(x_1, x_1) and Q(x_2, x_2) can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \text{ (where } \lambda \text{ is a parameter)}$$

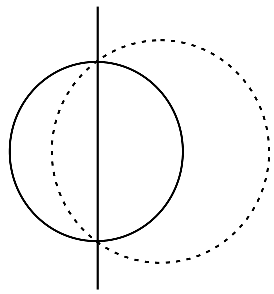


Figure 9.54

In this equation, $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is the equation of the circle with P and Q as the end points of the diameter and $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is the equation of the line through P and Q.

(e) The equation of the family of circles touching the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at point P(x_1, y_1) is $x^2 + y^2 + 2gx + 2fy + c + \lambda\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\} = 0$ or, $S + \lambda L = 0$, where, $L = 0$ is the equation of the tangent to the circle at P(x_1, y_1) and $\lambda \in R$.

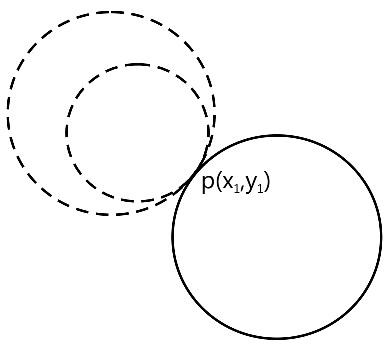


Figure 9.55

(f) The equation of family of circle, which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is

$(x - x_1)^2 + (y - y_1)^2 + \lambda\{(y - y_1) - m(x - x_1)\} = 0$. And if m is infinite, the family of circle is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$, (where λ is a parameter)

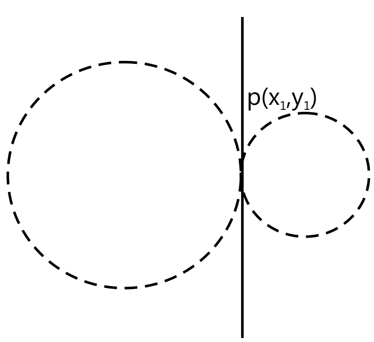


Figure 9.56

Note that $(x - x_1)^2 + (y - y_1)^2 = 0$ represents the equation of a point circle with centre at (x_1, y_1)

(g) Equation of the circles given in diagram is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) \pm \cot\theta \{(x - x_1)(y - y_2) - (x - x_2)(y - y_1)\} = 0$

(h) Family of circles circumscribing a triangle whose side are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by $L_1 L_2 + \lambda L_2 L_3 + \mu L_3 L_1 = 0$ provided coefficient of $xy = 0$ and co-coefficient of x^2 = co-efficient of y^2.

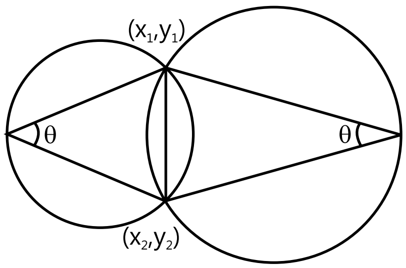


Figure 9.57

(i) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1 L_3 + \lambda L_2 L_4 = 0$ where value of λ can be found out by using condition that co-efficient of $x^2 = y^2$ and co-efficient of $xy = 0$.

Illustration 36: Find the equation of circle through the points A(1, 1) & B(2, 2) and whose radius is 1. **(JEE MAIN)**

Sol: As we know that, equation of family of circle passing through (x_1, y_1) and (x_2, y_2) is given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda(x - y) = 0$

Equation of AB is $x - y = 0$

$\therefore$ Equation of the family of circle passing through A and B is

$(x - 1)(x - 2) + (y - 1)(y - 2) + \lambda(x - y) = 0$ or $x^2 + y^2 + (\lambda - 3)x - (\lambda + 3)y + 4 = 0$

$\therefore$ Radius $= \sqrt{\frac{(\lambda-3)^2}{4} + \frac{(\lambda+3)^2}{4} - 4}$.

According to the question, $\sqrt{\frac{(\lambda-3)^2}{4} + \frac{(\lambda+3)^2}{4} - 4} = 1$

or $(\lambda - 3)^2 + (\lambda + 3)^2 - 16 = 4$ or $2\lambda^2 = 2$ or $\lambda = \pm 1$

$\therefore$ Equation of circle is $x^2 + y^2 - 2x - 4y + 4 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$

Illustration 37: Find the equations of circles which touches $2x - y + 3 = 0$ and pass through the points of intersection of the line $x + 2y - 1 = 0$ and the circle $x^2 + y^2 - 2x + 1 = 0$. **(JEE MAIN)**

Sol: Here in this problem the equation of family of circle will be $S + \lambda L = 0$ by solving this equation we will get centre and radius of required circle in the form of λ and as this circle touches the line $2x - y + 3 = 0$ hence perpendicular distance from centre of circle to the line is equal to the radius of circle.

Let the equation of the family of circles be $S + \lambda L = 0$.

$\therefore x^2 + y^2 - 2x + 1 + \lambda(x + 2y - 1) = 0$ or, $x^2 + y^2 - x(2 - \lambda) + 2\lambda y + (1 - \lambda) = 0$ ... (i)

$\Rightarrow$ Centre $(-g, -f)$ is $(\{2 - \lambda\}/2, -\lambda)$

$\Rightarrow r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{(2-\lambda)^2}{4} + \lambda^2 - (1-\lambda)} = \frac{1}{2}\sqrt{5\lambda^2} = \frac{\lambda}{2}\sqrt{5}$.

Since the circle touches the line 2x – y + 3 = 0,

$$\therefore \left|\frac{2.\left[(2-\lambda)/2\right]-(-\lambda)+3}{\pm\sqrt{5}}\right| = \frac{\lambda}{2}\sqrt{5} \text{ or, } 5 = \pm\frac{\lambda}{2}5 \quad \Rightarrow \lambda = \pm 2 \qquad \text{... (ii)}$$

Hence, the required circles are $x^2 + y^2 + 4y - 1 = 0$ and $x^2 + y^2 - 4x - 4y + 3 = 0$.

Illustration 38: If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y + p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for **(JEE MAIN)**

(A) All except two values of p (B) Exactly one value of p

(C) All values of p (D) All except one value of p.

Sol: (D) Here in this problem the equation of family of circle will be $S + \lambda L = 0$. and as the circle passes through (1, 1), we can find the values of P such that λ is any real no. except – 1.

Equation of a circle passing through P and Q is

$x^2 + y^2 + 3x + 7y + 2p - 5 + \lambda (x^2 + y^2 + 2x + 2y - p^2) = 0$ ... (i)

Since (i) also passes through (1 , 1), we get $(7 + 2p) - \lambda (p^2 - 6) = 0$

$$\Rightarrow \lambda = \frac{7+2p}{p^2-6} \neq -1 \Rightarrow p \neq -1.$$

Illustration 39: C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0) respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Equation of the common tangent to C_1 and C_3 which does not pass through C_2 is **(JEE ADVANCED)**

(A) $x - \sqrt{3}y + 2 = 0$ (B) $\sqrt{3}x - y + 2 = 0$

(C) $\sqrt{3}x - y - 2 = 0$ (D) $x + \sqrt{3}y + 2 = 0$

Sol: (B) Equation of any circle passing through any two point (x_1, y_1) and (x_2, y_2) is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

Equation of any circle passing through the centre of C_1 and C_2 is

$$(x - 0)(x - 1) + (y - 0)(y - 0) + \lambda\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow x^2 + y^2 - x + \lambda y = 0.$ ... (i)

If (i) represents C_3, its radius = 1

$\Rightarrow 1 = (1/4) + (\lambda^2/4) \qquad \Rightarrow \lambda = -\sqrt{3}$ (as λ cannot be +ve)

Hence, the equation of C_3 is $x^2 + y^2 - x - \sqrt{3}y = 0$.

Since the radius of C_1 and C_3 are equal, their common tangents will be parallel to the line joining their centres (0, 0)

and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

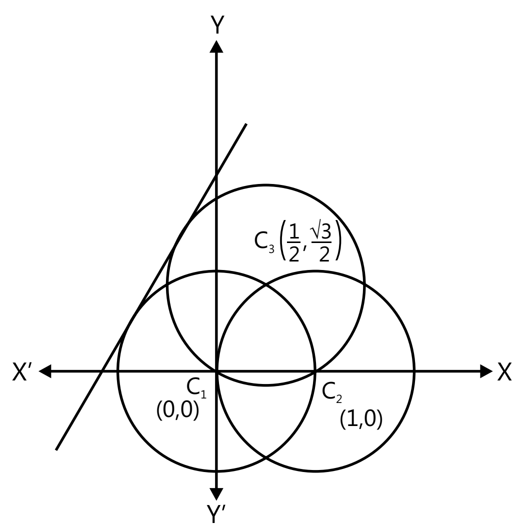


Figure 9.58

So, let the equation of a common tangent be $y = \sqrt{3}x + k$. ... (ii)

From the condition of tangency on C_1, we get $\left|\dfrac{k}{\sqrt{3+1}}\right| = 1 \Rightarrow k = \pm 2$

Since the tangent does not pass through C_2, the equation of the required common tangent is $\sqrt{3}x - y + 2 = 0$.

Illustration 40: Find the equation of circle circumscribing the triangle whose sides are $3x - y - 9 = 0$, $5x - 3y - 23 = 0$ & $x + y - 3 = 0$. **(JEE ADVANCED)**

Sol: Given $L_1 \equiv 3x - y - 9 = 0$ $L_2 \equiv 5x - 3y - 23 = 0$ $L_3 \equiv x + y - 3 = 0$

Family of circles circumscribing a triangle whose side are $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is

$L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of xy = 0 & co-coefficient of x^2 = co-efficient of y^2.

$\therefore L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0$

$\Rightarrow (3x - y - 9)(5x - 3y - 23) + \lambda(5x - 3y - 23)(x + y - 3) + \mu(3x - y - 9)(x + y - 3) = 0$

$\Rightarrow (15x^2 + 3y^2 - 14xy - 114x + 50y + 207) + \lambda(5x^2 - 3y^2 + 2xy - 38x - 14y + 69)$

$+ \mu(3x^2 - y^2 + 2xy - 18x - 6y + 27) = 0$

$\Rightarrow (5\lambda + 3\mu + 15)x^2 + (3 - 3\lambda - \mu)y^2 + xy(2\lambda + 2\mu - 14) - x(114 + 38\lambda + 18\mu)$

$+ y(50 - 14\lambda - 6\mu) + (207 + 69\lambda + 27\mu) = 0$ ... (i)

The equation (i) represents a circle if

coefficient of x^2 = coefficient of y^2

$\Rightarrow 5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu \Rightarrow 8\lambda + 4\mu + 12 = 0$; $2\lambda + \mu + 3 = 0$ ... (ii)

and, coefficient of xy = 0

$\Rightarrow 2\lambda + 2\mu - 14 = 0 \quad \Rightarrow \lambda + \mu - 7 = 0$ ... (iii)

From equation (ii) and (iii), we have $\lambda = -10$, $\mu = 17$

Putting these values of λ & μ in equation (i), we get $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

16. RADICAL AXIS AND RADICAL CENTRE

16.1 Radical Axis

The radical axis of two circles is defined as the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal. The radical axis of two circles is a straight line.

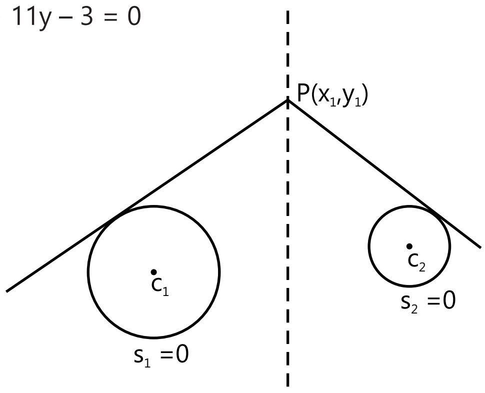


Figure 9.59

$\sqrt{S_1} = \sqrt{S_2}$

$\Rightarrow S_1 = S_2 \Rightarrow S_1 - S_2 = 0$

Consider two circles given by $S_1 = 0$ and $S_2 = 0$. Then the equation of the radical axis of the two circle is $S_1 - S_2 = 0$

i.e. $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$,

which is a straight line.

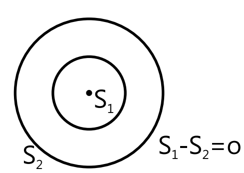


Figure 9.60

Properties of the radical axis

(a) For two intersecting circles the radical axis and common chord are identical. Also, the radical axis and the common tangent are same for two circles touching each other.

(b) The radical axis is perpendicular to the line joining the centres of the two circles.

(c) If two circles cut a third circle orthogonally, the radical axis of the two circles will pass through the centre of the third circle.

(d) Radical axis does not exist if circles are concentric.

(e) Radical axis does not always pass through the mid-point of the line joining the centre of the two circles.

(f) The radical axis of two circles bisects all common tangents of the two circles.

16.2 Radical Centre

The point of intersection of the radical axis of three circles, taken in pairs, is known as their radical centre.

Let the three circles be

$S_1 = 0$ .....(i), $S_2 = 0$ ...(ii), $S_3 = 0$ ...(iii)

Refer to the diagrams shown alongside.

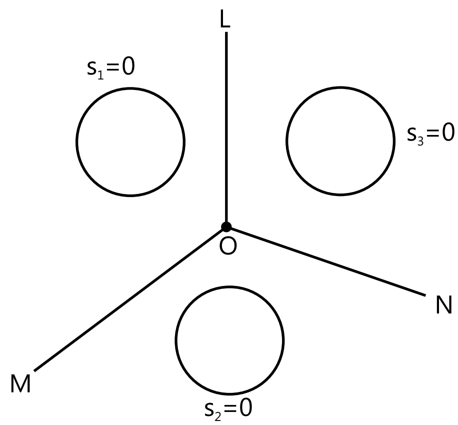


Figure 9.61

Let the straight line OL be the radical axis of the circles $S_1 = 0$ & $S_3 = 0$ and the straight line OM be the radical axis of the circles $S_1 = 0$ & $S_2 = 0$. The equation of any straight line passing through O is given by $(S_1 - S_2) + \lambda(S_3 - S_1) = 0$, where λ is any constant.

For $\lambda = 1$, this equation become $S_2 - S_3 = 0$, which is, equation of ON.

Clearly, the third radical axis also passes through the point where the straight lines OL and OM meet. Hence, the point of intersection of the three radical axis, O is the radical centre.

Properties of radical center

(a) Co-ordinates of radical centre can be found by solving the equation $S_1 = S_2 = S_3$.

(b) The radical centre does not exist if the centre of three circles are collinear.

(c) The circles with centre at radical centre and radius is equal to the length of tangents from radical centre to any of the circle will cut the three circle orthogonally.

(d) If circles are drawn on three sides of a triangle as diameter then radical centre of the three circles is the orthocenter of the triangle. Hence, in case of a right angled triangle, the radical centre of the three circles with the sides as diameter is the vertex with the right angle.

PLANCESS CONCEPTS

Alternate approach to find the equation of the tangent of a circle passing through a point lying on a given circle.

Consider a point (x_1, y_1) on the given circle $S_1 = 0$. Then the equation of a point circle with (x_1, y_1) as the centre is $S_2 \equiv (x - x_1)^2 + (y - y_1)^2 = 0$. Now we have two circles - one given circle and another point circle. We now have to find the radical axis of those two circles, which is $S_1 - S_2 = 0$.

E.g.: Given a circle $x^2 + y^2 = 8$ and the point on circle is (2, 2), we need to find equation of a tangent to the circle at point (2, 2).

Point circle: $(x - 2)^2 + (y - 2)^2 = 0 \quad \Rightarrow x^2 + y^2 - 4x - 4y + 8 = 0$

Hence, the radical axis is $S_1 - S_2 = 0$. $\quad \Rightarrow x + y = 4$, which is also the tangent to the given circle at the point (2, 2).

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 41: Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal. $3x^2 + 3y^2 + 4x - 6y - 1 = 0$, $2x^2 + 2y^2 - 3x - 2y - 4 = 0$, $2x^2 + 2y^2 - x + y - 1 = 0$. **(JEE MAIN)**

Sol: Here by using formula $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, and $S_3 - S_1 = 0$ we will get equations of radical axis and solving these equations we will get required co-ordinate.

Reducing the equation of the circles to the standard form,

$$S_1 \equiv x^2 + y^2 + \frac{4}{3}x - 2y - \frac{1}{3} = 0$$

$$S_2 \equiv x^2 + y^2 - \frac{3}{2}x - y - 2 = 0$$

$$S_3 \equiv x^2 + y^2 - \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2} = 0$$

Hence, the equations of the three radical axis is given by

$$L_1 \equiv \frac{17}{6}x - y + \frac{5}{3} = 0 \qquad \text{... (i)}$$

$$L_2 \equiv -x - \frac{3}{2}y - \frac{3}{2} = 0, \qquad \text{... (ii)}$$

$$\text{and, } L_3 \equiv -\frac{11}{6}x + \frac{5}{2}y + \frac{1}{6} = 0. \qquad \text{... (iii)}$$

Solving (i) and (ii), we get the point $\left(-\frac{16}{21}, \frac{31}{63}\right)$, which also satisfies the equation (iii).

This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

Illustration 42: Find the equation of the circle orthogonal to the three circles $x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 + 7x - 9y + 29 = 0$ **(JEE ADVANCED)**

Sol: By using formula of radical axis we will get co- ordinate of radical centre which is also equal to the centre of required circle.

The given circles are

$$S_1 \equiv x^2 + y^2 - 2x + 3y - 7 = 0 \qquad \text{... (i)}$$

$$S_2 \equiv x^2 + y^2 + 5x - 5y + 9 = 0 \qquad \text{... (ii)}$$

$$\text{and } S_3 \equiv x^2 + y^2 + 7x - 9y + 29 = 0 \qquad \text{... (iii)}$$

The radical axis of $S_1 = 0$ and $S_2 = 0$ is $7x - 8y + 16 = 0$ ... (iv)

The radical axis of $S_2 = 0$ and $S_3 = 0$ is $x - 2y + 10 = 0$ ...(v)

$\therefore$ The radical centre is (8, 9).

Therefore, the length of the tangent from (8, 9) to each of the given circles is $\sqrt{149}$.

$\therefore$ The required equation is $(x - 8)^2 + (y - 9)^2 = 149$ or $x^2 + y^2 - 16x - 18y - 4 = 0$.

Illustration 43: If two circles intersect a third circle orthogonally. Prove that their radical axis passes through the centre of the third circle. **(JEE ADVANCED)**

Sol: By considering equation of these circles as $S_r = x^2 + y^2 + 2g_rx + 2f_ry + c_r = 0$

(r = 1, 2, 3) and using radical axis formula we will prove given problem.

Let the given circles be $S_r = x^2 + y^2 + 2g_rx + 2f_ry + c_r = 0$ (r = 1, 2, 3)

Let S_1 and S_2 cut each other orthogonally, then we have

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$ ... (i)

Similarly, let S_2 and S_3 cut each other orthogonally, then we have

$2g_2g_3 + 2f_2f_3 = c_2 + c_3$ ... (ii)

Subtracting (ii) from (i), we get $2(g_1 - g_3)g_2 + 2(f_1 - f_3)f_2 = c_1 - c_3$ ... (iii)

Now the radical axis of S_1 and S_3 is $2(g_1 - g_3)x + 2(f_1 - f_3)y + c_1 - c_3 = 0$ ... (iv)

From (iii) and (iv), the point $(-g_2, -f_2)$ lies on the line (iv). Hence, proved.

Illustration 44: Prove that the square of the length of tangent that can be drawn from any point on one circle to another circle is equal to twice the product of the perpendicular distance of the point from the radical axis of the two circles, and the distance between their centres. **(JEE ADVANCED)**

Sol: Consider two circle as $S_1 \equiv x^2 + y^2 = a^2$ and $S_2 \equiv (x - h)^2 + y^2 = b^2$ and then by using radical axis formula and perpendicular distance formula we will prove given problem.

We have to prove that $PQ^2 = 2 \times PN \times C_1C_2$

Let the equation of the two circles be

$S_1 \equiv x^2 + y^2 = a^2$, and ... (i)

$S_2 \equiv (x - h)^2 + y^2 = b^2$ ... (ii)

Let $P \equiv (a\cos\theta, a\sin\theta)$ be a point on the circle $S_1 = 0$ $\quad \therefore PQ = \sqrt{a^2 - b^2 + h^2 - 2ah\cos\theta}$

and, Radical axis is $\{x^2 + y^2 - a^2\} - \{(x - h)^2 + y^2 - b^2\} = 0$

$\Rightarrow -2hx + h^2 + a^2 - b^2 = 0 \quad$ or $\quad x = \dfrac{h^2 + a^2 - b^2}{2h}$

$\Rightarrow PN = \dfrac{h^2 + a^2 - b^2}{2h} - a\cos\theta \quad = \dfrac{h^2 + a^2 - b^2 - 2ah\cos\theta}{2h}$

$\Rightarrow PN \times C_1C_2 = \dfrac{h^2 + a^2 - b^2 - 2ah\cos\theta}{2h} \times h \quad \Rightarrow PN \times C_1C_2 = \dfrac{PQ^2}{2}$

$\therefore PQ^2 = 2\,PN \times C_1C_2$

17. CO-AXIAL SYSTEM OF CIRCLES

A system (or a family) of circles, every pair of which have the same radical axis, are called co-axial circles.

(1) The equation of a system of co-axial circles, when the equation of the radical axis is $P \equiv lx + my + n = 0$ and, one circle of the system is $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ respectively, is $S + \lambda P = 0$ (λ is an arbitrary constant).

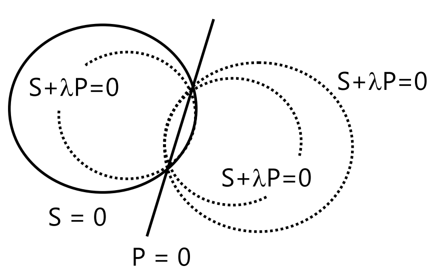


Figure 9.62

(2) The equation of a co-axial system of circles, when the equation of any two circles of the system are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ respectively, is $S_1 + \lambda(S_1 - S_2) = 0$

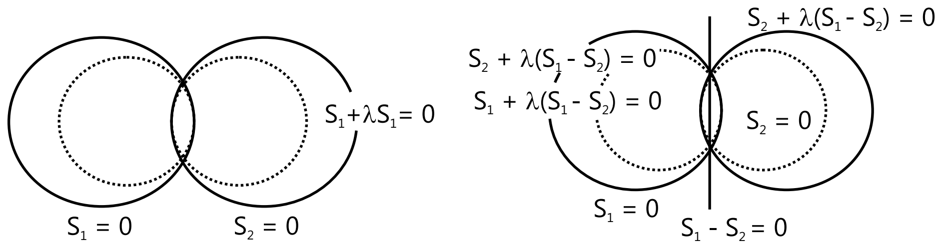


Figure 9.63

or $\quad S_2 + \lambda_1(S_1 - S_2) = 0$

Other form

$S_1 + \lambda S_2 = 0, \quad (\lambda \neq -1)$

Properties of co-axial System of Circles

(a) Centres of all circles of a coaxial system lie on a straight line which is perpendicular to the common radical axis as the line joining the centres of two circles is perpendicular to their radical axis.

(b) Circles passing through two fixed points P and Q form a coaxial system, because every pair of circles has the same common chord PQ and therefore, the same radical axis which is perpendicular bisector of PQ.

PLANCESS CONCEPTS

The equation of a system of co-axial circles in the simplest form is $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant. This is the system with center on x-axis and y-axis as common radical axis.

Anvit Tawar (JEE 2009, AIR 9)

Illustration 45: Find the equation of the system of coaxial circles that are tangent at $\left(\sqrt{2},4\right)$ to the locus of point of intersection of mutually perpendicular tangents to the circle $x^2 + y^2 = 9$. **(JEE ADVANCED)**

Sol: The locus of point of intersection of mutually perpendicular tangents is known as the Director circle. Hence by using formula of director circle and co-axial system of circle we will get required result.

$\therefore$ The equation of the locus of point of intersection of perpendicular tangents is

$$x^2 + y^2 = 18 \qquad \text{... (i)}$$

Since, $\left(\sqrt{2},4\right)$ satisfies the equation $x^2 + y^2= 18$,

$\therefore$ The tangent at $\left(\sqrt{2},4\right)$ to the circle $x^2 + y^2 = 18$ is $\quad x\cdot\sqrt{2} + y\cdot 4 = 18$ ... (ii)

The equation of the family of circles touching (i) at $\left(\sqrt{2},4\right)$ is

$$x^2 + y^2 - 18 + \lambda(\sqrt{2}x + 4y - 18) = 0 \text{ or } \quad x^2 + y^2 + \sqrt{2}\lambda x + 4\lambda y - 18(\lambda + 1) = 0 \qquad \text{... (iii)}$$

Also any two circles of (iii) have the same radical axis $\sqrt{2}x + 4y - 18 = 0$

$\therefore$ The required equation of coaxial circles is (iii).

Illustration 46: Find equation of circle co-axial with $S_1=x^2+y^2+4x+2y+1=0$ and $S_2 = 2x^2 + 2y^2 - 2x - 4y - 3 = 0$ and centre of circle lies on radical axis of these 2 circles. **(JEE MAIN)**

Sol: By using $S_1 - S_2 = 0$ and $S_1 + \lambda L = 0$ we will get equation of radical axis and equation of co-axial system of circle respectively.

$$S_1 - S_2 = 0 \qquad \Rightarrow 5x + \frac{5}{2} + 4y = 0 \qquad \Rightarrow 10x + 8y + 5 = 0$$

$\therefore$ The equation of the radical axis is $10x + 8y + 5 = 0$

The equation of the coaxial system of circles is $x^2 + y^2 + 4x + 2y + 1 + \lambda(10x + 5 + 8y) = 0$

$\Rightarrow$ Centre $\equiv [-(2+5\lambda), -(1+4\lambda)]$ which lies on radical axis, after substituting we get $\Rightarrow \lambda = -\frac{23}{82}$

Illustration 47: For what values of l and m the circles $5(x^2 + y^2) + ly - m = 0$ belongs to the coaxial system determined by the circles $x^2 + y^2 + 2x + 4y - 6 = 0$ and $2(x^2 + y^2) - x = 0$? **(JEE ADVANCED)**

Sol: By using radical axis formula i.e. $S_1 - S_2 = 0$ we will get equations of radical axis and by solving them simultaneously we will get required value of l and m.

Let the circles be $S_1 \equiv x^2 + y^2 + 2x + 4y - 6 = 0;$

$$S_2 \equiv x^2 + y^2 - \frac{1}{2}x = 0;$$

$$S_3 \equiv x^2 + y^2 + \frac{l}{5}y - \frac{m}{5} = 0.$$

The equation of the radical axis of circles $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$,

i.e., $$x^2 + y^2 + 2x + 4y - 6 - \left(x^2 + y^2 - \frac{1}{2}x\right) = 0 \qquad \text{or,} \qquad 5x + 8y - 12 = 0 \qquad \text{... (i)}$$

The equation of the radical axis of circles $S_2 = 0$ and $S_3 = 0$ is $S_2 - S_3 = 0$,

i.e., $$x^2 + y^2 - \frac{1}{2}x - \left(x^2 + y^2 + \frac{l}{5}y - \frac{m}{5}\right) = 0 \qquad \text{or,} \qquad 5x + 2ly - 2m = 0 \qquad \text{... (ii)}$$

On comparing (i) and (ii), $\frac{5}{5} = \frac{8}{2l} = \frac{-12}{-2m} \Rightarrow 1 = \frac{4}{l} = \frac{6}{m}$ $\qquad \therefore l = 4, m = 6.$

18. LIMITING POINTS

Limiting point of system of co-axial circles are the centres of the point circles belonging to the family

Let the circle be $x^2 + y^2 + 2gx + c = 0$ where g is a variable and c is a constant.

$\therefore$ Centre $\equiv (-g, 0)$ and Radius $= \sqrt{g^2 - c}$.

A circle is said to be a point circle, if the radius is equal to 0, i.e. $\sqrt{g^2 - c} = 0 \Rightarrow g = \pm\sqrt{c}$

Thus, we get the two limiting points of the given co-axial system as $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

Depending on the sign of c, either the limiting points are real and distinct, real and coincident or imaginary.

18.1 System of Co-axial Circles when Limiting Points are given

Let (a, b) and (α, β) be two limiting points of a coaxial system of circles. Then, the equation of the corresponding point circles are $S_1 \equiv (x - a)^2 + (y - b)^2 = 0$ and $S_2 \equiv (x - \alpha)^2 + (y - \beta)^2 = 0$.

$\therefore$ The coaxial system of circles is given by $S_1 + \lambda S_2 = 0, \lambda \neq -1$.

or, $(x - a)^2 + (y - b)^2 + \lambda\{(x - \alpha)^2 + (y - \beta)^2\} = 0, \lambda \neq -1$.

PLANCESS CONCEPTS

If origin is a limiting point of the coaxial system containing the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ then the other limiting point is $\left(\frac{-gc}{g^2+f^2}, \frac{-fc}{g^2+f^2}\right)$.

A common tangent drawn to any two circles of a coaxial system subtends an angle of $\frac{\pi}{2}$ at the limiting points.

Chinmay S Purandare (JEE 2012, AIR 698)

Illustration 48: Equation of a circles through the origin and belonging to the co-axial system, of which the limiting points are (1, 2), (4, 3) is **(JEE ADVANCED)**

(A) $x^2 + y^2 - 2x + 4y = 0$ (B) $x^2 + y^2 - 8x - 6y = 0$

(C) $2x^2 + 2y^2 - x - 7y = 0$ (D) $x^2 + y^2 - 6x - 10y = 0$

Sol: (C) As we know, if (a, b) and (α, β) be two limiting points of a coaxial system of circles. Then, the equation of the corresponding point circles are $S_1 \equiv (x-a)^2 + (y-b)^2 = 0$ and $S_2 \equiv (x-\alpha)^2 + (y-\beta)^2 = 0$ so by using the formula of co-axial system i.e. $S_1 + \lambda S_2 = 0$ we will get required result.

Equations of the point circles having (1, 2) and (4,3) as centres is

$S_1 \equiv (x-1)^2 + (y-2)^2 = 0 \quad \Rightarrow x^2 + y^2 - 2x - 4y + 5 = 0$

and, $S_2 \equiv (x-4)^2 + (y-3)^2 = 0 \quad \Rightarrow x^2 + y^2 - 8x - 6y + 25 = 0$

$\therefore$ The co-axial system of circles is $S_1 + \lambda S_2 = 0$.

i.e. $x^2 + y^2 - 2x - 4y + 5 + \lambda(x^2 + y^2 - 8x - 6y + 25) = 0$ ... (i)

If (0, 0) lies on the circle given by equation (i), then

$0^2 + 0^2 - 2(0) - 4(0) + 5 + \lambda(0^2 + 0^2 - 8(0) - 6(0) + 25) = 0$

$\Rightarrow 5 + 25\lambda = 0 \quad$ or, $\lambda = -\left(\frac{1}{5}\right)$.

$\therefore$ The equation of the required circle is $5(x^2 + y^2 - 2x - 4y + 5) - (x^2 + y^2 - 8x - 6y + 25) = 0$

$\Rightarrow 4x^2 + 4y^2 - 2x - 14y = 0 \quad \Rightarrow 2x^2 + 2y^2 - x - 7y = 0$.

19. IMAGE OF THE CIRCLE BY LINE MIRROR

Here, let us consider a general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and a line $lx + my + n = 0$. If we take the image of the circle in the given line, then the radius of image circle remains unchanged and the centre lies on the opposite side of the line at an equal distance.

Let the centre of image circle be (x_1, y_1).

$\therefore$ Slope of $C_1C_2 \times$ Slope of $(lx + my + n = 0) = -1$ ..... (i)

And the mid-point of C_1 and C_2 lies on the line $lx + my + n = 0$

$$l\left(\frac{x_1-g}{2}\right)+m\left(\frac{y_1-f}{2}\right)+n=0 \quad \text{..... (ii)}$$

From (i) and (ii), we get the centre of the image circle and the radius is $\sqrt{\left(g^2+f^2-c\right)}$ (same as the given circle), and hence the equation of the image.

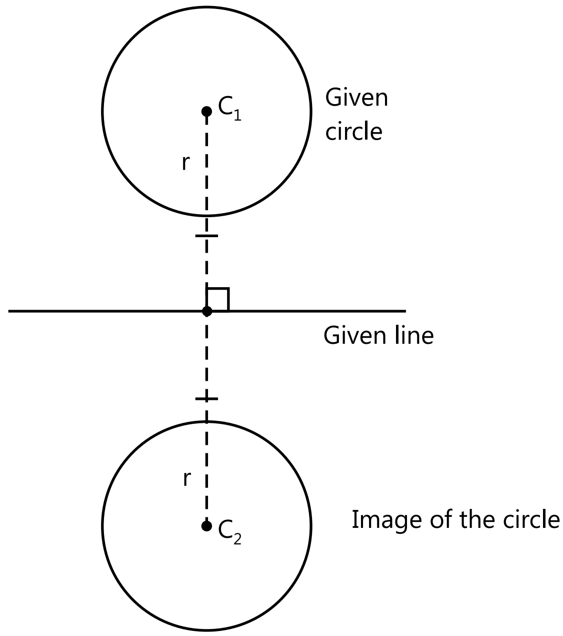


Figure 9.64: Image of a circle

PROBLEM-SOLVING TACTICS

(a) Let S = 0, S′ = 0 be two circles with centers C_1, C_2 and radii R_1, R_2 respectively.

(i) If $C_1C_2 > r_1 + r_2$ then each circle lies completely outside the other circle.

(ii) If $C_1C_2 = r_1 + r_2$ then the two circles touch each other externally. (Trick) the point of contact divides C_1C_2 in the ratio $r_1 : r_2$ internally.

(iii) If $|r_1 - r_2| < C_1C_2 < r1 + r2$ then the two circles intersect at two points P and Q.

(iv) If $C_1C_2 = |r_1 - r_2|$ then the two circles touch each other internally. (Trick) The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ externally.

(v) If $C_1C_2 < |r_1 - r_2|$ then one circle lies completely inside the other circle.

(b) Two intersecting circles are said to cut each other orthogonally if the angle between the circles is a right angle. Let the circles be $S = x^2 + y^2 + 2gx + 2fy + c = 0$, $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$.

And let d be the distance between the centers of two intersecting circles with radii r1, r2. The two circles will intersect orthogonally if and only if

(i) $D^2 =$ and

(ii) $2g\,g' + 2f\,f' = c + c'$.

FORMULAE SHEET

1. General equation of a circle: $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) Centre of the circle = $(-g, -f)$.

$g = \frac{1}{2}$ coefficient of x, and $f = \frac{1}{2}$ coefficient of y.

(ii) $r = \sqrt{g^2 + f^2 - c}$

2 $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if

(i) $a = b \neq 0$ (ii) $h = 0$ (iii) $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$ (iv) $g^2 + f^2 - c \geq 0$

3. if centre of circle is (h, k) and radius 'r' then equation of circle is: $(x - h)^2 + (y - k)^2 = r^2$

4. The equation of the circle drawn on the straight line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is : $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Centre : $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. $r = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$

5. (i) In parametric form:

$x = -g + \sqrt{(g^2 + f^2 - c)}\cos\theta$ and $y = -f + \sqrt{(g^2 + f^2 - c)}\sin\theta$, $(0 \leq \theta < 2\pi)$

6. (i) Circle passing through three non-collinear points

$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ is represented by $\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$

7. Circle circumscribing the triangle formed by the lines

$a_i x + b_i y + c_i = 0 \ (i = 1,2,3)$: $\begin{vmatrix} \frac{a_1^2 + b_1^2}{a_1 x + b_1 y + c_1} & a_1 & b_1 \\ \frac{a_2^2 + b_2^2}{a_2 x + b_2 y + c_2} & a_2 & b_2 \\ \frac{a_3^2 + b_3^2}{a_3 x + b_3 y + c_3} & a_3 & b_3 \end{vmatrix} = 0$

8. Intercepts length made by the circle On X and Y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively.

9. Position of point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.

When $S_1 \equiv x_1^2 + y_1^2 + 2gx + 2fy + c >\ =\ < 0$ respectively.

10. The power of $P(x_1, y_1)$ w.r.t. $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to PA. PB which is $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_2 + c$.

PA. PB = PC.PD = PT^2 = square of the length of a tangent

11. Intercept length cut off from the line y =mx + c by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\dfrac{a^2\left(1+m^2\right)-c^2}{1+m^2}}$

12. The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

13. The equation of tangent at $(a \cos\theta, a \sin\theta)$ is $x \cos\theta + y \sin\theta = a$

14. Condition for tangency:

line $y = mx + c$ is tangent of the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$

and the point of contact of tangent $y = mx \pm a\sqrt{1+m^2}$ is $\left(\dfrac{\mp ma}{\sqrt{1+m^2}}, \dfrac{\pm a}{\sqrt{1+m^2}}\right)$

15. The length of the tangent from a point $P(x_1,y_1)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

16. Pair of tangent from point (0, 0) to the circle are at right angles if $g^2 + f^2 = 2c$.

17. Equation of director circle of the circle $x^2 + y^2 = a^2$ is equal to $x^2 + y^2 = 2a^2$.

18. Equation of Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

19. The equation of normal at any point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xy_1 - x_1y = 0$ or $\dfrac{x}{x_1} = \dfrac{y}{y_1}$.

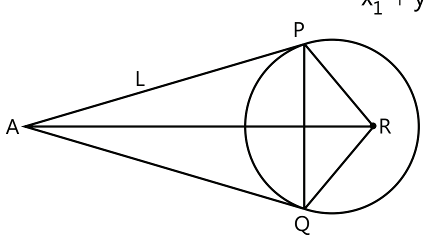
20. Equation of normal at $(a \cos\theta, a \sin\theta)$ is $y = x \tan\theta$ or $y = mx$.

21. The equation of the chord of contact of tangents drawn from a point (x_1, y_1)

to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_2 = a^2$. And

to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

22. Area of Δ APQ is given by $\dfrac{a\left(x_1^2 + y_1^2 - a^2\right)^{\frac{3}{2}}}{x_1^2 + y_1^2} = \dfrac{RL^3}{R^2 + L^2}$. Where L & R are length of tangent and radius of circle.



23. The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Bisected at the point (x_1, y_1) is given $T = S_1$.

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

24. The equation of the common chord of two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is equal to $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ i.e., $S_1 - S_2 = 0$.

25. Length of the common chord : $PQ = 2(PM) = 2\sqrt{C_1P^2 - C_1M^2}$. Where,

C_1P = radius of the circle $S_1 = 0$

C_1M =perpendicular length from the centre C_1 to the common chord PQ.

26. Equation of polar of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 = a^2$

w.r.t. (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ and $xx_1 + yy_1 - a^2 = 0$. Respectively.

27. The pole of the line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 = a^2$: $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$

28. P (x_1, y_1) and $Q(x_2, y_2)$ are conjugate points of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

When $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

If P and Q are conjugate points w.r.t. a circle with centre at O and radius r then $PQ^2 = OP^2 + OQ - 2r^2$.

29. The points P and T are a intersection point of direct common tangents and transverse. Common tangents respectively, and it divide line joining the centres of the circles externally and internally respectively in the ratio of their radii.

$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2} \text{ (externally)}$$

$$\frac{C_1T}{C_2T} = \frac{r_1}{r_2} \text{ (internally)}$$

Hence, the ordinates of P and T are.

$$P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) \text{ and } T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

30. If two circles $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ of r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos(180 - \theta) = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \quad \text{or} \quad \cos = (180 - \theta) = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_1^2 + f_1^2 - c_2}}.$$

31. Condition for orthogonality: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

32. $S_1 - S_2 = 0$ the equation of the radical axis of the two circle. i.e. $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ which is a straight line.

33. The two limiting points of the given co-axial system are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

34. If two limiting points of a coaxial system of circles is (a, b) and (α, β).

then $S_1 + \lambda S_2 = 0$, $\lambda \neq -1$. or, $\{(x - a)^2 + (y - b)^2\} + \lambda\{(x - \alpha)^2 + (y - \beta)^2\} = 0$, $\lambda \neq -1$ is the

Coaxial system of circle.

35. If origin is a limiting point of the coaxial system containing the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then the other limiting point is $\left(\frac{-gc}{g^2 + f^2}, \frac{-fc}{g^2 + f^2}\right)$.

Solved Examples

JEE Main/Boards

Example 1: The given curves $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + c = 0$ intersect each other at four concyclic points then prove that $\frac{a-b}{h} = \frac{A-B}{H}$.

Sol: Equation of second degree curve passing through the intersections of the given curves is $S_1 + \lambda S_2 = 0$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda(Ax^2 + 2Hxy + By^2 + 2Gx + Fy + C) = 0$ ..... (i)

intersection points of of the two curves are concyclic,

(i) must be a circle for some λ.

$\therefore$ Coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

$\therefore a + \lambda A = b + \lambda B$

and $2h + \lambda \cdot 2H = 0$ or $a - b = \lambda(B - A)$

and $h = -\lambda H$

$\therefore \quad \frac{a-b}{h} = \frac{\lambda(B-A)}{-\lambda H}$; $\therefore \frac{a-b}{h} = \frac{A-B}{H}$.

Example 2: Find the equation of a circle which cuts the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ orthogonally and which passes though (3, 0) and touches the y-axis.

Sol: When two circle intersects each other orthogonally then $2(g_1g_2 + f_1f_2) = c_1 + c_2$. Hence by considering centre as (h, k) and using given condition we can solve problem.

Let C(h, k) be the centre of required circle

radius of circle = $\sqrt{(h-3)^2 + k^2}$ = |h|

$\therefore \quad (h - 3)^2 + k^2 = h^2$

or $\quad k^2 - 6h + 9 = 0$ .....(i)

Required circle is $(x - h)^2 + (y - k)^2 = h^2$

or $\quad x^2 + y^2 - 2hx - 2ky + k^2 = 0$

It is intersected by $x^2 + y^2 - 6x + 4y - 3 = 0$, orthogonally;

$\therefore \quad 2(-3)(-h) + 2(2)(-k) = k^2 - 3$

or $\quad 6h - 4k + 3 = k^2$ .....(ii)

Solve (i) and (ii) : h = 3, k = 3

Required circle is $x^2 + y^2 - 6x - 6y + 9 = 0$

Example 3: Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a circle C_1 (of diameter 6). If centre of C_1 lies in the first quadrant, find concentric circle C_2 which cuts intercepts of length 8 units on each given line.

Sol: Consider centre of required circle is (h, k) and by using perpendicular distance formula from centre to given tangent we will get value of h and k.

Let centre of circle C_1 be O (h, k), where h > 0 and k > 0

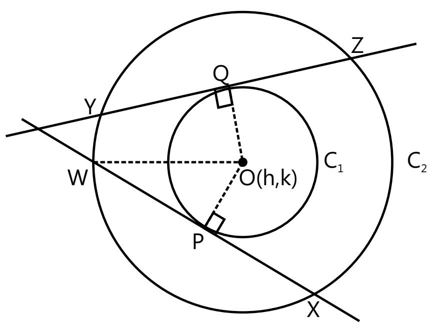
WP = 4 and OP = 3 ... (given)

In $\triangle$OWP,

$$OW = \sqrt{OP^2 + WP^2} = \sqrt{4^2 + 3^2} = 5$$

OW = 5 = radius of C_2.

$$\Rightarrow \frac{|5h + 12k - 10|}{13} = \frac{|5h - 12k - 40|}{13} = 3$$



$$\frac{5h + 12k - 10}{13} = \pm\left(\frac{5h - 12k - 40}{13}\right)$$

$\Rightarrow$ Either $5h + 12k - 10 = 5h - 12k - 40$

$\Rightarrow$ 24 k = - 30

$\Rightarrow k = \frac{-30}{40}$ (Not possible)

Or $5h + 12k - 10 = -5h + 12k + 40$

$\Rightarrow$ 10 h = 50

$\Rightarrow$ H = 5

Substituting h = 5 in

$$\frac{|5h + 12k - 10|}{13} = 3$$

$\Rightarrow k = 2$ (as $k > 0$)

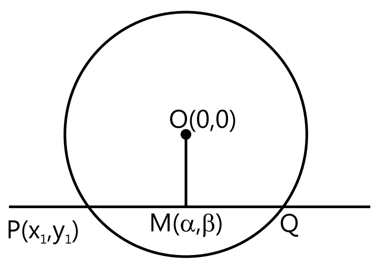
Hence, equation of required circle is

$$(x - 5)^2 + (y - 2)^2 = 25$$

Example 4: Find the locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which pass through a given point (x_1, y_1).

Sol: As line joining centre of given circle to the mid point of chord is perpendicular to the chord and hence product of their slope will be – 1. Therefore by considering mid point of chord as (α, β) and by finding their slope we will get required equation.

Let M (α, β) be the middle point of any chord PQ through the given point (x_1, y_1). The centre of the circle is O (0, 0). Clearly MO is perpendicular to PQ.



Now, slope of PQ = $\dfrac{\beta - y_1}{\alpha - x_1}$

slope of OM = $\dfrac{\beta - 0}{\alpha - 0} = \dfrac{\beta}{\alpha}$

$\therefore \quad \dfrac{\beta - y_1}{\alpha - x_1} \cdot \dfrac{\beta}{\alpha} = -1$

or $\quad \alpha(\alpha - x_1) + \beta(\beta - y_1) = 0$

$\therefore$ the equation of the locus of M (α, β) is

$x(x - x_1) + y(y - y_1) = 0$

Alternative

The equation of chord when mid-point is known is $T = S_1$

Let the mid-point be (α, β)

$\therefore x\,\alpha + y\,\beta - \cancel{a^2} = \alpha^2 + \beta^2 - \cancel{a^2}$

$\because$ It passes through (x_1, y_1) we get

$x, \alpha + y, \beta = \alpha^2 + \beta^2$

$\Rightarrow \alpha(\alpha - x_1) + \beta(\beta - y_1) = 0$

$\therefore$ Required locus is

$x(x - x_1) + y(y - y_1) = 0$

Example 5: From a point P tangents are drawn to circles $x^2 + y^2 + x - 3 = 0$,

$x^2 + y^2 - \left(\dfrac{5}{3}\right)x + y = 0$ and $4x^2 + 4y^2 + 8x + 7y + 9 = 0$,

and they are of equal lengths. Find equation of a circle passing through P and touching the line $x + y = 5$ at A(6, –1).

Sol: By reading the problem we get that P is a radical centre of these circles. Hence by radical axis formula we can obtain co-ordinate of point P, as required circle is passing from these points so we can obtain required equation.

Write third circle as

$x^2 + y^2 + 2x + \left(\dfrac{7}{4}\right)y + \left(\dfrac{9}{4}\right) = 0$

By definition, P is radical centre of three circles. Equation of two of the radical axis are

$\left(\dfrac{8}{3}\right)x - y - 3 = 0$ and $x + \left(\dfrac{7}{4}\right)y + \left(\dfrac{21}{4}\right) = 0$

which intersect at P(0, –3). Let required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

with centre Q(–g, –f)

P(0, –3) lies on it

$\Rightarrow \quad -6f + c + 9 = 0 \qquad \text{... (i)}$

A(6, –1) lies on it

$\Rightarrow \quad 12g - 2f + c + 37 = 0 \qquad \text{... (ii)}$

Since, PA is perpendicular to $x + y = 5$

$\therefore \quad \left(\dfrac{-f+1}{-g-6}\right)(-1) = -1 \qquad \text{... (iii)}$

$\Rightarrow \quad f - g = 7$

Solving (i), (ii) and (iii) for f, g and c, we have

$f = \dfrac{7}{2}$, $g = -\dfrac{7}{2}$ and $c = 12$.

Hence equation of required circle is

$x^2 + y^2 - 7x + 7y + 12 = 0$.

Example 6: Find the equation of a circle which touches the line $x + y = 5$ at the point P(–2, 7) and cut the circle

$x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally.

Sol: Using the concept of family of circle and the condition for two circles to be orthogonal, we can find the equation of the required circle.

As the circle is touching the line $x + y = 5$. It (-2, 7).

Consider the equation of circle as

$$(x+2)^2+(y+7)^2+\lambda(x+y-5)=0$$

$$\Rightarrow x^2+y^2+x(4+\lambda)+y(\lambda-14)+53-5\lambda=0 \quad \text{....(i)}$$

$\therefore$ As the circle given equation (i) is orthogonal to

$x^2+y^2+4x-6y+9=0$,

We have

$$(4+\lambda).2+(\lambda-14)(-3)=53-5\lambda+9$$

$$\Rightarrow 8+2\lambda-3\lambda+42=62-5\lambda$$

$$\Rightarrow 4\lambda=12$$

$$\Rightarrow \lambda=3$$

$\therefore$ Equation of the circle is $x^2+y^2+7x-11y+38=0$.

Example 7: Find the equation of the circle described on the common chord of the circles $x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 + 8y + 7 = 0$ as diameter.

Sol: Use Geometry to find the centre and the radius of the required circle.

For $x^2+y^2-4x-5=0$

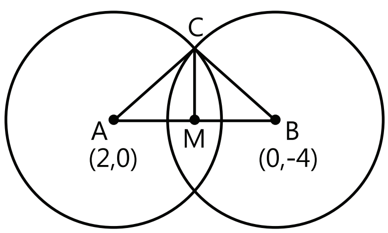
Centre $\equiv (+2, 0)$

Radius = 3

For $x^2+y^2+8y+7=0$

Centre $\equiv (0, -4)$

Radius = 3



The mid point of AB is the centre of the required circle

i.e. $M\equiv (1, -2)$

and Radius $=\sqrt{AC^2-AM^2}$

$=\sqrt{9-5}$

= 2

Equation of circle is $(x-1)^2+(y-2)^2=4$.

Example 8: Prove that, for all $c \in R$, the pole of the line $\frac{x}{a}+\frac{y}{b} = 1$ with respect to the circle $x^2 + y^2 = c^2$ lies on a fixed line.

Sol: As polar of point (x_1, y_1) with respect to the circle $x^2 + y^2 = c^2$ is same as line $\frac{x}{a}+\frac{y}{b}=1$.

On comparing the two equations, we can prove the given statement.

Let the pole be (x_1, y_1). Then the polar of (x_1,y_1) with respect to the circle $x^2 + y^2 = c^2$ is

$$xx_1 + yy_1 = c^2 \quad \text{... (i)}$$

Now, the line (i) and $\frac{x}{a}+\frac{y}{b} = 1$ must be the same line.

$\therefore$ comparing coefficients, $\frac{x_1}{1/a}=\frac{y_1}{1/b}=\frac{c^2}{1}$

or $\quad ax_1=by_1=c^2$,

$\therefore \quad ax_1=by_1$

$\therefore$ (x_1, y_1) always lies on the line ax = by which is a fixed line.

Example 9: Inside the circle $x^2 + y^2 = a^2$ is inscribed an equilateral triangle with the vertex at (a , 0). The equation of the side opposite to this vertex is

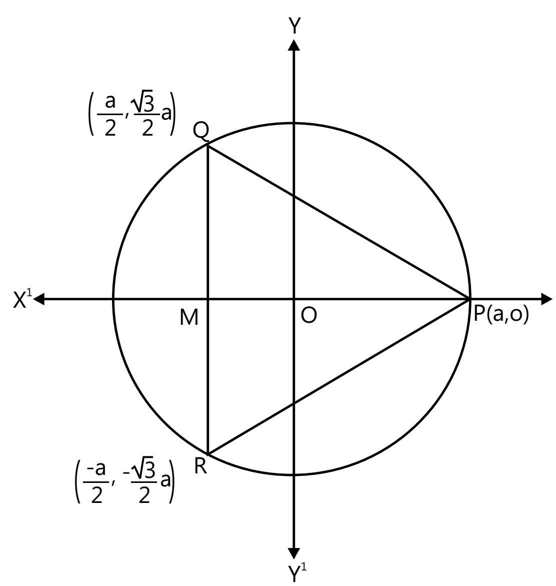
(A) $2x - a = 0$ (B) $x + a = 0$

(C) $2x + a = 0$ (D) $3x - 2a = 0$

Sol: (C) As P (a , 0) be the vertex of the equilateral triangles PQR inscribed in the circle $x^2 + y^2 = a^2$ Let M be the middle point of the side QR, then MOP is perpendicular to QR and O being the centroid of the triangle OP = 2 (OM).

(Circumcentre and Centroid of an equilateral triangle are same)

So if (h, k) be the coordinates of M, then



$\frac{2h+a}{3} = 0$ and $\frac{2k+0}{3} = 0$

$\Rightarrow h = -\left(\frac{a}{2}\right)$ and $k = 0$

and hence the equation of BC is

$x = -\frac{a}{2}$ or $2x + a = 0$.

Example 10 : Find the radical centre of the three circles $x^2 + y^2 = a^2$, $(x - c)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = a^2$.

Sol: Here by using the formula

$S_1 - S_2 = 0$, $S_2 - S_3 = 0$ and $S_3 - S_1 = 0$

we will get equation of radical axis and by solving them we can obtain requird radical centre.

Radical axis of first & second circle is given by

$(x^2 + y^2) - (x^2 + y^2 - 2cx + c^2) = 0$

or $\quad x = \frac{c}{2}$

Also the radical axis of first and third circle is given by

$(x^2 + y^2) - (x^2 + y^2 - 2by + b^2) = 0$

or $\quad y = \frac{b}{2}$

$\Rightarrow$ The radical centre = $\left(\frac{c}{2}, \frac{b}{2}\right)$.

JEE Advanced/Boards

Example 1: Two distinct chords drawn from the point P(a, b) to the circle $x^2 + y^2 - ax - by = 0$, $(ab \neq 0)$, are bisected by the x-axis. Show that $a^2 > 8b^2$.

Sol: As Circle passes through (0, 0) and P(a, b) .Consider the chord PQ intersect x-axis at A; then, Q is $(\alpha, -b)$. Hence by substituting this point to given equation of circle we can solve above problem.

$\therefore \alpha^2 + b^2 - a\alpha + b^2 = 0$ or $\alpha^2 - a\alpha + 2b^2 = 0$

Hence, Discriminant > 0

$\Rightarrow a^2 > 8b^2$

Example 2: Let T_1, T_2 be two tangents drawn from (–2 , 0) to the circle C : $x^2 + y^2 = 1$. Determine circles touching C and having T_1, T_2 as their pair of tangents. Further find the equation of all possible common tangents to these circles, when taken two at time.

Sol: As we know Equation of any tangent to $x^2 + y^2 = 1$, is $y = mx \pm \sqrt{1+m^2}$ and perpendicular distance from centre to tangent is equal to its radius. By using this condition we can solve above problem.

As they are drawn from A(–2, 0), conditions are

$0 = -2m \pm \sqrt{1+m^2}$

$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$

Equations of tangents become

$T_1 : \sqrt{3}y = x + 2$

$T_2 : \sqrt{3}y = -x - 2$

Circles touching C and having T_1 and T_2 as tangents must have their center on x-axis (the angle bisector of T_1 and T_2).

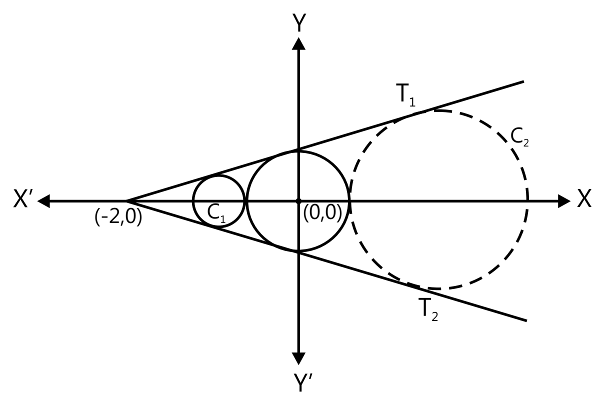
Let C_1 and C_2 be the 2 circles and M $(h_1, 0)$ & L$(h_2, 0)$ be their respective centers where

$h_1 > 0$ and $h_2 < 0$

By tangency of T_1, perpendicular distance from centre M is equal to radius r_1 of the circle C_1

$\therefore r_1 = \frac{h_1 + 2}{2}$

As C_1 and C touch each other $r_1 = h_1 - 1$



or $\quad \frac{h_1 + 2}{2} + 1 = h_1$

or $\quad h_1 = 4$

$\therefore$ For circle C_1 : centre is M(4, 0) and radius = 3.

Similarly for circle C_2, $-h_2 - 1 = \left|\frac{h_2 + 2}{2}\right|$

$\Rightarrow \quad -2h_2 - 2 = h_2 + 2$

($\therefore h_2 > -2$; see figure)

$\Rightarrow \quad -3h_2 = 4$

or $\quad h_2 = -\frac{4}{3}$ and radius = $\frac{1}{3}$.

Equations of two circles are $(x-4)^2 + y^2 = 9$ and

$$\left(x+\frac{4}{3}\right)^2 + y^2 = \frac{1}{9}$$

C_1 & C have x = 1 as transverse common tangent and C_2 & C have x = –1 as transverse common tangent.

Example 3 : Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre O. Show that the centroid of the triangle PAB as P moves on the circle is a circle.

Sol: By considering point P(r cos θ , r sinθ) and centroid as point (x_1, y_1) we can obtain required result.

ΔOAB is isosceles with

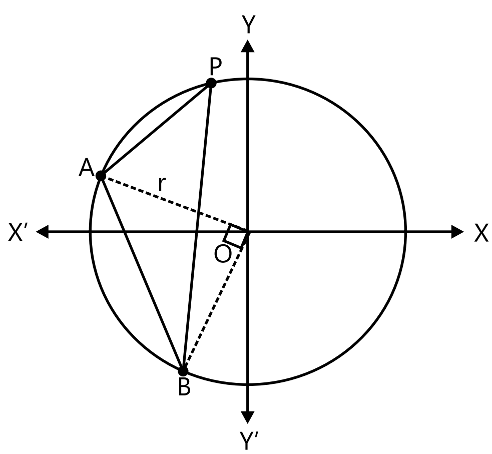
OA = OB = x(say)

We may assume AB is parallel to and below x-axis

$\therefore \quad x^2 + x^2 = r^2 \quad \Rightarrow \quad x = \dfrac{r}{\sqrt{2}}$

$\therefore$ B is $\left(\dfrac{r}{\sqrt{2}}, -\dfrac{r}{\sqrt{2}}\right)$ and A is $\left(-\dfrac{r}{\sqrt{2}}, -\dfrac{r}{\sqrt{2}}\right)$

Let P be (r cos θ , r sinθ) and centroid of ΔPAB be G (x_1, y_1)



$$\therefore x_1 = \frac{r\cos\theta + \frac{r}{\sqrt{2}} - \frac{r}{\sqrt{2}}}{3},$$

$$y_1 = \frac{r\sin\theta - \frac{r}{\sqrt{2}} - \frac{r}{\sqrt{2}}}{3}$$

$3x_1 = r\cos\theta$; $3y_1 = r(\sin\theta - \sqrt{2})$

Eliminating θ, we get

$$\therefore \quad \left(\frac{3x_1}{r}\right)^2 + \left(\frac{3y_1}{r} + \sqrt{2}\right)^2 = 1$$

or $\quad x_1^2 + \left(y_1 + \dfrac{\sqrt{2}r}{3}\right)^2 = \dfrac{r^2}{9}$

$\therefore$ Locus of (x_1, y_1) is a circle.

Example 4: Derive the equation of the circle passing through the point P(2, 8) and touches the lines 4x – 3y – 24 = 0 and 4x + 3y – 42 = 0 and coordinates of the centre less than or equal to 8.

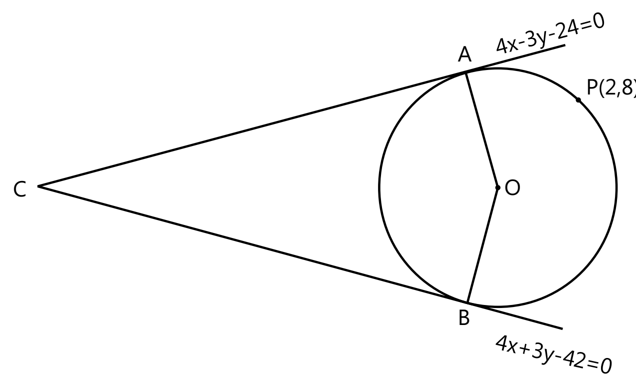
Sol: Here using Equations of bisectors of angle between the lines we will get co-ordinate of centre of circle i.e O. and as OA = OP we can obtain required equation of circle. consider O is the center of circle.

Let $L_1 \equiv 4x - 3y - 24 = 0$

$L_2 \equiv 4x + 3y - 42 = 0$

and Let A and B denote the respective points of contact

Equations of bisectors of angle between the lines are;



$$\frac{4x-3y-24}{5} = \pm\frac{4x+3y-42}{5}$$

i.e., $y = 3 \quad \& \quad x = \dfrac{33}{4}$

Since O lies on one of these bisectors and x-coordinate of O is less then or equal to 8,

$\therefore$ O lies on y = 3.

Let O be (a,3). Then, OA = CP

or $\quad \left(\dfrac{4a-33}{5}\right)^2 = (a-2)^2 + 25$

or $\quad 16a^2 - 264a + (33)^2 = 25\{a^2 - 4a + 29\}$

or $\quad 9a^2 + 164a - 364 = 0$

or $\quad (a-2)(9a+182) = 0$

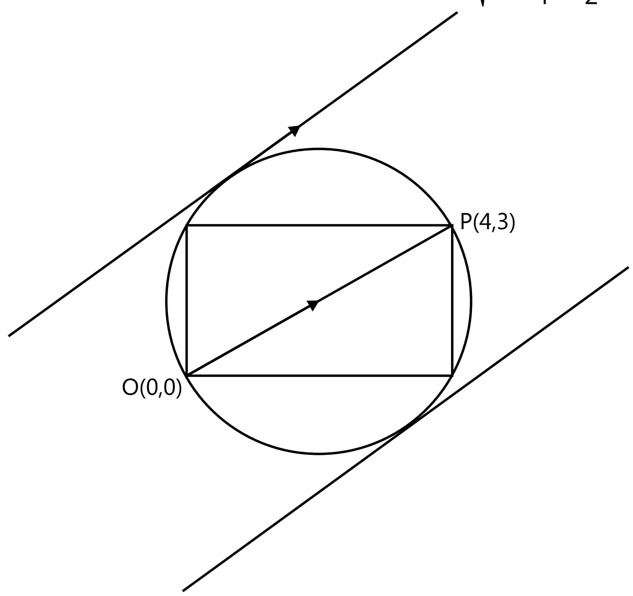
$\therefore \quad a = 2$ or $\quad a = -\dfrac{182}{9}$

and radius = OP.

Example 5: Coordinates of a diagonal of a rectangle are (0, 0) and (4, 3). Find the equations of the tangents to the circumcircle of the rectangle which are parallel to this diagonal.

Sol: Here centre of circle is the mid-point of line OP hence by using slope point form we can get required equation of tangents.

Two extremities are O (0, 0) and P (4, 3). Middle point of the diagonal OP is $M\left(2,\frac{3}{2}\right)$ which is the centre of the circumscribed circle and radius is OM = $\sqrt{4+\frac{9}{4}}=\frac{5}{2}$



A line parallel to OP is $y = \frac{3}{4}x + c$

It is a tangent to the circumscribed circle.

Therefore length of perpendicular from

$M\left(2,\frac{3}{2}\right)$ to it = $\frac{5}{2} \Rightarrow \frac{\left|\frac{3}{4}(2)-\frac{3}{2}+C\right|}{\sqrt{1+\frac{9}{16}}}=\frac{5}{2}$

or $\quad C=\pm\frac{5}{2}\cdot\frac{5}{4}=\pm\frac{25}{8}$

Hence tangents are $y=\frac{3}{4}x\pm\frac{25}{8}$

or $\quad 3x-4y\pm\frac{25}{2}=0.$

Example 6: The equations two circles are $x^2 + y^2 + \lambda x + c = 0$ and $x^2 + y^2 + \mu x + c = 0$. Prove that one of the circles will be within the other if $\lambda\mu > 0$ and $c > 0$.

Sol: The condition for one circle to be within the other is

$C_1C_2 < |r_1 - r_2|$

Without the loss of generality,

Let $\lambda > \mu$

$\therefore C_1\,C_2 < r_1 - r_2 \Rightarrow \left(\frac{\lambda-\mu}{2}\right) < \sqrt{\frac{\lambda^2}{4}-c} - \sqrt{\frac{\mu^2}{4}-c}$

$\Rightarrow \frac{\lambda^2}{4}+\frac{\mu^2}{4}-2\times\frac{\lambda}{2}\times\frac{\mu}{2} < \frac{\lambda^2}{4}-c+\frac{\mu^2}{4}-c$

$-2\sqrt{\left(\frac{\lambda^2}{4}-c\right)\left(\frac{\mu^2}{4}-c\right)}$

$\Rightarrow 2\sqrt{\left(\frac{\lambda^2}{4}-c\right)\left(\frac{\mu^2}{4}-c\right)} < 2.\frac{\lambda\mu}{4}-2c$

$\frac{\lambda^2\mu^2}{16}-c\left(\frac{\lambda^2}{4}+\frac{\mu^2}{4}\right)+c^2 < \frac{\lambda^2\mu^2}{16}+c^2 -$

$2\times c\times\frac{\lambda\mu}{4}$

$c\left(\frac{\lambda^2}{4}+\frac{\mu^2}{4}-2\times\frac{\lambda}{2}\times\frac{\mu}{2}\right) > 0$

$c\left(\frac{\lambda}{2}-\frac{\mu}{2}\right)^2 > 0$

$\Rightarrow C > 0$

Also $\therefore \lambda > \mu$

$\left(\frac{-\mu}{2},0\right)$ will be inside

$x^2+y^2+\lambda x+c=0$

$\Rightarrow \frac{\mu^2}{4}+0-\frac{\mu\lambda}{2}+c<0$

$\because \frac{\mu^2}{4}+c>0$

$\therefore \frac{\lambda\,\mu}{2} > \frac{\mu^2}{4}+c$

$\therefore \frac{\lambda\,\mu}{2} > 0$

$\Rightarrow \lambda\,\mu > 0$

Hence, proved.

Example 7: A circle touches the line y = x at a point P such that OP = $4\sqrt{2}$ where O is the origin. The circle contains the point (–10, 2) in its interior and the length of its chord on the line x + y = 0 is $6\sqrt{2}$. Find the equation of the circle.

Sol: In this question, the concept of rotation of axes would be useful.

Let the new co-ordinate axis be rotated by an angle of 45° in the clockwise direction. Then

$X = x\ \cos\ (\theta) + \ y\ \sin(\theta)$

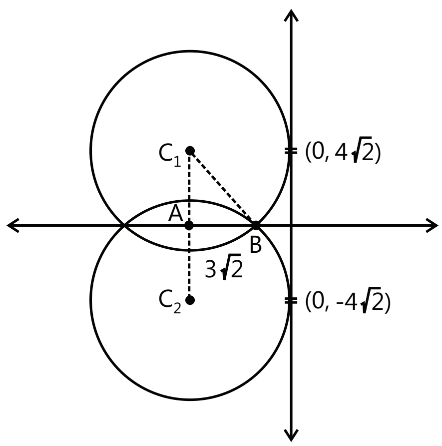
$Y = -x\ \ \sin\ \ (\theta) + \ y\ \cos(\theta)$

Where $\theta = 45°$

$\therefore\ X\ = \dfrac{x-y}{\sqrt{2}}$

$Y\ = \dfrac{x+y}{\sqrt{2}}$

The image after rotation would be



In Δ ABC, AC, $= 4\sqrt{2}$

AB $= 3\sqrt{2}$

$\therefore$ Radius = $\sqrt{\left(4\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2} = b\sqrt{2}$

$\therefore$ Equation of the circle is

$$\left(X+5\sqrt{2}\right)^2 + \left(Y \mp 4\sqrt{2}\right)^2 = \left(5\sqrt{2}\right)^2$$

$$\text{Or, } \left(\frac{x-y}{\sqrt{2}} + 5\sqrt{2}\right)^2 + \left(\frac{x+y}{\sqrt{2}} \mp 4\sqrt{2}\right)^2 = \left(5\sqrt{2}\right)$$

$$\text{or, } \left(x-y+10\right)^2 + \left(x+y\pm 8\right)^2 = 100$$

But, since (-10, 2) lies inside the circle.

The equation of the circle is

$$\left(x-y+10\right)^2 + \left(x+y+8\right)^2 = 100$$

$$\text{Or, } x^2 + y^2 + 100 - 2xy - 20y + 20x$$

$$+x^2 + y^2 + 64 + 2xy + 16y + 16x = 100$$

$$\text{Or, } 2x^2 + 2y^2 + 36x - 4y + 64 = 0$$

$$\text{Or, } x^2 + y^2 + 18x - 2y + 32 = 0$$

Example 8: Derive the equation of the circle passing through the centres of the three given circles $x^2 + y^2 - 4y - 5 = 0$,

$x^2 + y^2 + 12x + 4y + 31 = 0$ and

$x^2 + y^2 + 8x + 10y + 32 = 0$.

Sol: Find the relation between the centres of the circle and there use the appropriate form of circle.

Let P, Q and R denote the centres of the given circle

$P \equiv (0, 2)$, $Q = (-6, -2)$ and

$R \equiv\ (-4, -5)$

$$\therefore\ m_{PQ} = \frac{-2-2}{-6-0} = \frac{-4}{-6} = \frac{2}{3}$$

$$m_{QR} = \frac{-5+2}{-4+6} = \frac{-3}{2}$$

$$\therefore\ m_{PQ}\,.\,m_{QR} = \frac{2}{3} \times \frac{-3}{2} = -1$$

$\Rightarrow$ PQ is perpendicular to QR

$\therefore$ Using diameter form, we get

$(x - 0)(x + 4) + (y - 2)(y + 5) = 0$

Example 9: Area of Quadrilateral PQRS is 18, side PQ || RS and PQ = 2RS and PS $\perp$ PQ and RS. then radius a circle drawn inside the quadrilateral PQRS touching all the sides is,

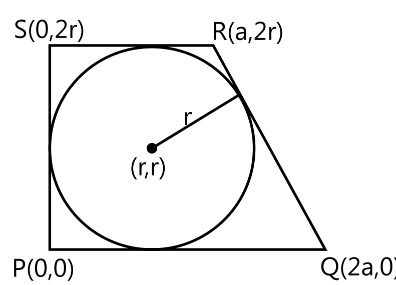
(A) 3 (B) 2 (C) $\dfrac{3}{2}$ (D) 1

Sol: (B) Let r be the radius of the circle, then PS = 2r. Let P be the origin and PQ and PS as x-axis and y-axis respectively.

$\therefore$ The coordinates of P, Q, R, S are (0, 0), (2a, 0), (a, 2r) and (0, 2r) respectively.

$\therefore$ Area (PQRS) = $\left(\dfrac{1}{2}\right)(a + 2a)(2r) = 18$

$\Rightarrow\quad ar = 6.$



$\therefore$ Equation of QR is

$$\left(y-y_1\right) = \left(\frac{y_2-y_1}{x_2-x_1}\right)\left(x-x_1\right)$$

$\Rightarrow (y-2r) = \left(\dfrac{0-2r}{2a-a}\right)(x-a)$

$\Rightarrow (y-2r) = \dfrac{-2r}{a}(x-a)$

$\Rightarrow ay - 2ar = -2rx + 2ar$

$\Rightarrow 2rx + ay - 4ar = 0$

$\therefore$ QR is a tangent to the circle

$\therefore \left|\dfrac{2r^2 + ar - 4ar}{\sqrt{4r^2+a^2}}\right| = r$

$\Rightarrow \left|\dfrac{r(2r-3a)}{\sqrt{4r^2+a^2}}\right| = r$

$\Rightarrow (2r-3a)^2 = 4r^2 + a^2$

$\Rightarrow 4r^2 + 9a^2 - 12\ ar = 4r^2 + a^2$

$\Rightarrow 8a^2 = 12\ ar$

$\Rightarrow 2a^2 = 3\ ar$

$\Rightarrow 2a^2 = 3 \times 6$

$\Rightarrow a = 3$

$\therefore r = 2\ (\because ar = 6)$

Example 10: A circle having centre at (0, 0) and radius equal to 'a' meets the x - axis at P and Q. A(α) and B(β) are points on this circle such that $\alpha - \beta = 2\gamma$, where γ is a constant. Then locus of the point of intersection of PA and QB is

(A) $x^2 - y^2 - 2ay \tan\gamma = a^2$

(B) $x^2 + y^2 - 2ay \tan\gamma = a^2$

(C) $x^2 + y^2 + 2ay \tan\gamma = a^2$

(D) $x^2 - y^2 + 2ay \tan\gamma = a^2$

Sol: (B) Let the equation of the circle be $x^2 + y^2 = a^2$

$\therefore P \equiv (-a, 0)$ and $Q = (a, 0)$

$\therefore$ Equation of PA is

$(y-0) = \dfrac{a\sin\alpha - 0}{a\cos\alpha + a}(x+a)$

$\Rightarrow y = \dfrac{a\ \sin\alpha}{a(\cos\alpha + 1)}(x+a)$

$\Rightarrow y = \dfrac{a.2\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2}}{a.2\cos^2\frac{\alpha}{2}}(x+a)$

$\Rightarrow y = \tan\dfrac{\alpha}{2}(x+a)$ ...(i)

Similarly, equation of BQ is

$(y-v) = \dfrac{a\sin\beta - 0}{a\cos\beta - a}(x-a)$

$\Rightarrow y = \dfrac{2\ a.\sin\frac{\beta}{2}.\cos\frac{\beta}{2}}{-a \times 2\ \sin\frac{\beta}{2}}(x-a)$

$\Rightarrow y = -\cot\left(\dfrac{\beta}{2}\right)(x-a)$ ...(ii)

Now, we eliminate α, β using (i) and (ii)

$\because \alpha - \beta = 2r$

$\Rightarrow \dfrac{\alpha}{2} - \dfrac{\beta}{2} = r$

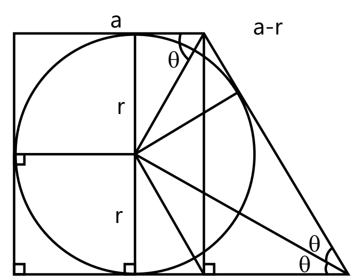
$\Rightarrow \tan\left(\dfrac{\alpha}{2} - \dfrac{\beta}{2}\right) = \tan r$

$\Rightarrow \dfrac{\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}}{1 + \tan\frac{\alpha}{2}.\tan\frac{\beta}{2}} = \tan r$

$\Rightarrow \dfrac{\frac{y}{x+a} - \frac{a-x}{y}}{1 + \frac{y}{a+x} \times \frac{a-x}{y}} = \tan\ \gamma$

$\Rightarrow \dfrac{y^2 - a^2 + x^2}{ay + xy + ay - xy} = \tan\gamma$

$\Rightarrow x^2 + y^2 - 2ay\ \tan\gamma - a^2 = 0$



JEE Main/Boards

Exercise 1

Q.1 Find the equation of the circle whose centre lies on the line $2x - y - 3 = 0$ and which passes through the points (3, –2) and (–2, 0).

Q.2 Show that four points (0, 0), (1, 1), (5, –5) and (6, –4) are concylic.

Q.3 Find the centre, the radius and the equation of the circle drawn on the line joining A(–1, 2) and B(3, –4) as diameter.

Q.4 Find the equation of the tangent and the normal to the circle $x^2 + y^2 = 25$ at the point P(–3, –4).

Q.5 Show that the tangent to $x^2 + y^2 = 5$ at (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$

Q.6 Find the equation of the tangents to the circle $x^2 + y^2 - 2x + 8y = 23$ drawn from an external point (8, –3).

Q.7 Find the equation of the circle whose centre is (–4, 2) and having the line $x - y = 3$ as a tangent

Q.8 Find the equation of the circle through the points of intersections of two given circles

$x^2 + y^2 - 8x - 2y + 7 = 0$ and

$x^2 + y^2 - 4x + 10y + 8 = 0$ and passing through (3, –3).

Q.9 Find the equation of chord of the circle $x^2 + y^2 - 4x = 0$ which is bisected at the point (1, 1).

Q.10 Find the equation of chord of contact of the circle $x^2 + y^2 - 4x = 0$ with respect to the point (6, 0).

Q.11 Find the length of the tangent drawn from the point (3, 2) to the circle $4x^2 + 4y^2 + 4x + 16y + 13 = 0$.

Q.12 Obtain the equations of common tangents of the circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 12x + 27 = 0$.

Q.13 The centres of the circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ are $\left(\frac{1}{2}, \pm\sqrt{2}\right)$.

Q.14 The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter.

Q.15 Show that the line $x + y = 2$ touches the circles $x^2 + y^2 = 2$ and $x^2 + y^2 + 3x + 3y - 8 = 0$ at the point where the two circles touch each other.

Q.16 One of the diameters of the circle circumscribing the rectangle ABCD is $4y = x + 7$. If A and B are the points (–3, 4) and (5, 4) respectively, find the area of the rectangle.

Q.17 A circle of radius 2 lies in the first quadrant and touches both the axes of co-ordinates, Find the equation of the circle with centre at (6, 5) and touching the above circle externally.

Q.18 If $\left(m_i, \frac{1}{m_i}\right)$; i = 1, 2, 3, 4 are four distinct point on a circle, show that $m_1m_2m_3m_4 = 1$.

Q.19 Show that the circle on the chord $x\cos\alpha + y\sin\alpha - p = 0$ of the circle $x^2 + y^2 = a^2$ as diameter is $x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha - p) = 0$.

Q.20 Find the length of the chord of the circle $x^2 + y^2 = 16$ which bisects the line joining the points (2, 3) and (1, 2) perpendicularly.

Q.21 Find the angle that the chord of circle $x^2 + y^2 - 4y = 0$ along the line $x + y = 1$ subtends at the circumference of the larger segment.

Q.22 Prove that the equation $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$, where λ is a parameter, represents a family of circles passing through two fixed points A and B on the x-axis. Also find the equation of that circle of the family, the tangents to which at A and B meet on the line $x + 2y + 5 = 0$.

Q.23 Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

Q.24 If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the co-ordinate axes in concyclic points, prove that $a_1a_2 = b_1b_2$.

Q.25 Show that the length of the tangent from any point on the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle

$x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is $\sqrt{c_1 - c}$.

Q.26 Find the point from which the tangents to the three circles $x^2 + y^2 - 4x + 7 = 0$,

$2x^2 + 2y^2 - 3x + 5y + 9 = 0$

and $x^2 + y^2 + y = 0$ are equal in length. Find also this length.

Q.27 The chord of contact of tangents from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that a, b, c are in G.P.

Q.28 Obtain the equation of the circle orthogonal to both the circles

$x^2 + y^2 + 3x - 5y + 6 = 0$ and

$4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4y + 1 = 0$.

Q.29 From the point A (0, 3) on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that AM = 2AB. Find the equation of the locus of M.

Q.30 From the origin, chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. Find the equation to the locus of the middle points of these chords.

Q.31 Tangent at any point on the circle $x^2 + y^2=a^2$ meets the circle $x^2 + y^2 = b^2$ at P and Q. Find the condition on a and b such that tangents at P and Q meet at right angles.

Q.32 The tangent from a point to the circle $x^2 + y^2 = 1$ is perpendicular to the tangent from the same point to the circle $x^2 + y^2 = 3$. Show that the locus of the point is a circle .

Q.33 A variable circle passes through the point A (a, b) and touches the x-axis. Show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.

Q.34 AB is a diameter of a circle. CD is a chord parallel to AB and 2CD = AB. The tangent at B meets the line AC (produced) at E. Prove that AE = 2AB.

Exercise 2

Single Correct Choice Type

Q.1 Centres of the three circles

$x^2 + y^2 - 4x - 6y - 14 = 0$

$x^2 + y^2 + 2x + 4y - 5 = 0$

and $x^2 + y^2 - 10x - 16y + 7 = 0$

(A) Are the vertices of a right triangle

(B) The vertices of an isosceles triangle which is not regular

(C) Vertices of a regular triangle

(D) Are collinear

Q.2 $2x^2 + 2y^2 + 2\lambda x + \lambda^2 = 0$ represents a circle for :

(A) Each real value of λ

(B) No real value of λ

(C) Positive λ

(D) Negative λ

Q.3 The area of an equilateral triangle inscribed in the circle $x^2 + y^2 - 2x = 0$ is

(A) $\frac{3\sqrt{3}}{4}$ (B) $\frac{3\sqrt{3}}{2}$

(C) $\frac{3\sqrt{3}}{8}$ (D) None of these

Q.4 A circle of radius 5 has its centre on the negative x-axis and passes through the point (2, 3). The intercept made by the circle on the y-axis is

(A)10 (B) $2\sqrt{21}$

(C) $2\sqrt{11}$ (D) imaginary y-intercept

Q.5 The radii of the circle $x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 6y = 6$ and $x^2 + y^2 - 4x - 12y = 9$ are in

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

Q.6 If the equation $x^2 + y^2 + 2\lambda x + 4 = 0$ and $x^2 + y^2 - 4\lambda y + 8 = 0$ represent real circles then the value of λ can be

(A) 5 (B) 2

(C) 3 (D) All of these

Q.7 The equation of the image of the circle $x^2 + y^2 +16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is;

(A) $x^2 + y^2 + 32x - 4y + 235 = 0$

(B) $x^2 + y^2 + 32x + 4y - 235 = 0$

(C) $x^2 + y^2 + 32x - 4y - 235 = 0$

(D) $x^2 + y^2 + 32x + 4y + 235 = 0$

Q.8 The circle described on the line joining the points (0, 1), (a, b) as diameter cuts the x-axis in points whose abscissae are roots of the equation :

(A) $x^2 + ax + b = 0$ (B) $x^2 - ax + b = 0$

(C) $x^2 + ax - b = 0$ (C) $x^2 - ax - b = 0$

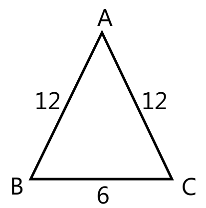
Q.9 A straight line l_1 with equation $x - 2y + 10 = 0$ meets the circle with equation $x^2 + y^2 = 100$ at B in the first quadrant. A line through B, perpendicular to l_1 cuts the y-axis at P (0, t). The value of 't' is

(A) 12 (B) 15 (C) 20 (D) 25

Q.10 If $\left(a, \frac{1}{a}\right), \left(b, \frac{1}{b}\right), \left(c, \frac{1}{c}\right)$ and $\left(d, \frac{1}{d}\right)$ are four distinct point on a circle of radius 4 units then, abcd is equal to

(A) 4 (B) $\frac{1}{4}$ (C) 1 (D) 16

Q.11 The radius of the circle passing through the vertices of the triangle ABC, is



(A) $\frac{8\sqrt{15}}{5}$ (B) $\frac{3\sqrt{15}}{5}$ (C) $3\sqrt{15}$ (D) $3\sqrt{2}$

Q.12 The points A(a, 0), B(0, b), C(c, 0) and D(0, d) are such that ac = bd and a, b, c, d are all non-zero. Then the points

(A) Form a parallelogram (B) Do not lie on a circle

(C) Form a trapezium (D) Are concyclic

Q.13 Four unit circles pass through the origin and have their centres on the coordinate axes. The area of the quadrilateral whose vertices are the points of intersection (in pairs) of the circle, is

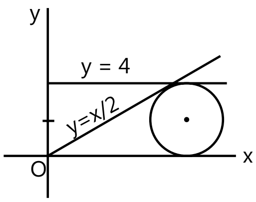
(A) 1 sq. unit

(B) $2\sqrt{2}$ sq. units

(C) 4sq. units

(D) Cannot be uniquely determined, insufficient data

Q.14 The x-coordinate of the center of the circle in the first quadrant (see figure) tangent to the lines $y = \frac{1}{2}x$, y= 4 and the x-axis is



(A) $4 + 2\sqrt{5}$ (B) $4 + \frac{8\sqrt{5}}{5}$

(C) $2 + \frac{6\sqrt{5}}{5}$ (D) $8 + 2\sqrt{5}$

Q.15 From the point A (0, 3) on the circle $x^2 + 4x + (y - 3)^2 = 0$ a chord AB is drawn and extended to a point M such that AM = 2AB. The equation of the locus of M is,

(A) $x^2 + 8x + y^2 = 0$

(B) $x^2 + 8x + (y - 3)^2 = 0$

(C) $(x - 3)^2 + 8x + y^2 = 0$

(D) $x^2 + 8x + 8y^2 = 0$

Q.16 If L_1 and L_2 are the length of the tangent from (0, 5) to the circles $x^2 + y^2 + 2x - 4 = 0$ and $x^2 + y^2 - y + 1 = 0$ then

(A) $L_1 = 2L_2$ (B) $L_2 = 2L_1$ (C) $L_1 = L_2$ (D) $L_1^2 = L_2$

Q.17 The line $2x - y + 1 = 0$ is tangent to the circle at the point (2, 5) and the centre of the circles lies on $x - 2y = 4$. The radius of the circle is

(A) $3\sqrt{5}$ (B) $5\sqrt{3}$ (C) $2\sqrt{5}$ (D) $5\sqrt{2}$

Q.18 Coordinates of the centre of the circle which bisects the circumferences of the circles $x^2 + y^2 = 1$; $x^2 + y^2 + 2x - 3 = 0$ and $x^2 + y^2 + 2y - 3 = 0$ is

(A) (–1, –1) (B) (3, 3) (C) (2, 2) (D) (–2, –2)

Q.19 The anglebetween the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

Q.20 In a right triangle ABC, right angled at A, on the leg AC as diameter, a semicircle is described. The chord joining A with the point of intersection of the hypotenuse and the semicircle, then the length AC equals to

(A) $\frac{AB \cdot AD}{\sqrt{AB^2 + AD^2}}$ (B) $\frac{AB \cdot AD}{AB + AD}$

(C) $\sqrt{AB \cdot AD}$ (D) $\frac{AB \cdot AD}{\sqrt{AB^2 - AD^2}}$

Q.21 Locus of all point P (x, y) satisfying $x^3 + y^3 + 3xy = 1$ consists of union of

(A) A line and an isolated point

(B) A line pair and an isolated point

(C) A line and a circle

(D) A circle and an isolated point.

Previous Years' Questions

Q.1 The circle passing through the point(–1, 0) and touching the y-axis at (0, 2) also passes through the point ***(2011)***

(A) $\left(-\frac{3}{2},0\right)$ (B) $\left(-\frac{5}{2},2\right)$ (C) $\left(-\frac{3}{2},\frac{5}{2}\right)$ (D) (–1, –4)

Q.2 Consider the two curves $C_1 : y^2 = 4x$

$C_2 : x^2 + y^2 - 6x + 1 = 0$, then ***(2008)***

(A) C_1 and C_2 touch each other only at one point

(B) C_1 and C_2 touch each other exactly at two points

(C) C_1 and C_2 intersect (but do not touch) at exactly two points

(D) C_1 and C_2 neither intersect nor touch each other

Q.3 If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre (2, 1), then the radius of the circle is ***(2004)***

(A) $\sqrt{3}$ (B) $\sqrt{2}$ (C) 3 (D) 2

Q.4 The centre of circle inscribed in square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$, is ***(2003)***

(A) (4, 7) (B) (7, 4) (C) (9, 4) (D) (4, 9)

Q.5 If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is ***(2002)***

(A) 4 (B) $2\sqrt{5}$ (C) 5 (D) $3\sqrt{5}$

Q.6 If the circle $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is ***(2000)***

(A) 2 or $-\frac{3}{2}$ (B) –2 or $-\frac{3}{2}$

(C) 2 or $\frac{3}{2}$ (D) –2 or $\frac{3}{2}$

Q.7 The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (–4, 3) respectively, then $\angle QPR$ is equal to ***(2002)***

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Q.8 The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is ***(1998)***

(A) 0 (B) 1 (C) 3 (D) 4

Q.9 Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$. ***(2007)***

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I.

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I.

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

Statement-I: The tangents are mutually perpendicular.

Statement-II: The locus of the points from which a mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.

Q.10 Find the equation of circle touching the line $2x + 3y + 1 = 0$ at the point (1, –1) and is orthogonal to the circle which has the line segment having end points (0, –1) and (–2, 3) as the diameter. ***(2004)***

Q.11 Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C ***(2001)***

Q.12 Consider the family of circles $x^2 + y^2 = r^2$, $2< r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordinate axis at A and B, then find the equation of the locus of the mid points of AB. ***(1999)***

Q.13 C_1 and C_2 are two concentric circle the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. ***(1998)***

Q.14 The length of the diameter of the circle which louches the x-axis at the point $(1,0)$ and passes through the point $(2,3)$ is ***(2012)***

(A) $\frac{10}{3}$ (B) $\frac{3}{5}$ (C) $\frac{6}{5}$ (D) $\frac{5}{3}$

Q.15 The circle through (1, –2) and touching the axis of x at (3,0) also passes through the point ***(2013)***

(A) $(2,-5)$ (B) $(5,-2)$ (C) $(-2,5)$ (D) $(-5,2)$

Q.16 The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, and having centre at $(0,3)$ is ***(2013)***

(A) $x^2+y^2-6y+7=0$ (B) $x^2+y^2-6y-5=0$
(C) $x^2+y^2-6y+5=0$ (D) $x^2+y^2-6y-7=0$

Q.17 let C be the circle with centre at (1,1) and radius =1. If T is the circle centred at $(0,y)$, passing through origin and touching the circle C externally, then radius of T is equal to ***(2014)***

(A) $\frac{\sqrt{3}}{\sqrt{2}}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Q.18 The number of common tangents to circles $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x-18y+26=0$, is ***(2015)***

(A) 2 (B) 3 (C) 4 (D) 1

Q.19 The centres of those circles which touch the circle, $x^2+y^2-8x-8y-4=0$, externally and also touch the x-axis, lie on: ***(2016)***

(A) An ellipse which is not a circle

(B) A hyperbola

(C) A parabola

(D) A circle

JEE Advanced/Boards

Exercise 1

Q.1 Let S : $x^2 + y^2 - 8x - 6y + 24 = 0$ be a circle and O is the origin. Let OAB is the line intersecting the circle at A and B. On the chord AB a point P is taken. The locus of the point P in each of the following cases.

(i) OP is the arithmetic mean of OA and OB

(ii) OP is the geometric mean of OA and OB

(iii) OP is the harmonic mean between OA and OB

Q.2 A circle $x^2 + y^2 + 4x - 2\sqrt{2}y + c = 0$ is the director circle of circle S_1 and S_1 is the director circle of circle S_2 and so on. If the sum of radii of all these circles is 2, then the value of c is equal to $\sqrt{n}$ where $n \in N$. Find the value of n.

Q.3 If the circle $x^2 + y^2 + 4x + 22y + a = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - b = 0$ (where a, b > 0), then find the maximum value of (ab).

Q.4 Real number x, y satisfies $x^2 + y^2 = 1$. If the maximum and minimum value of the expression $z = \frac{4-y}{7-x}$ are M and m respectively, then find the value (2M + 6m).

Q.5 The radical axis of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ and

$2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x - 2y + 1 = 0$. Show that either $g = \frac{3}{4}$ or f = 2

Q.6 Consider a family of circles passing through two fixed points A (3, 7) & B (6, 5). The chords in which the circle $x^2 + y^2 - 4x - 6y + 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

Q.7 Find the equation of circle passing through (1, 1) belonging to the system of co-axial circles that are tangent at (2, 2) to the locus of the point of intersection of mutually perpendicular tangent to the circle $x^2 + y^2 = 4$.

Q.8 The circle C : $x^2 + y^2 + kx + (1 + k)y - (k + 1) = 0$ passes through two fixed points for every real number k. Find

(i) the coordinates of these points.

(ii) the minimum value of the radius of a circle C.

Q.9 Find the equation of a circle which is co-axial with circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ and

$x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the centre of the circle to be determined lies on the radical axis of these two circles.

Q.10 Find the equation of the circle passing through the points of intersection of circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Q.11 The centre of the circles S = 0 lie on line $2x - 2y + 9 = 0$ & S = 0 cuts orthogonally the circle $x^2 + y^2 = 4$. Show that circle S = 0 passes through two fixed points & find their coordinates.

Q.12 Find the equation of a circle passing through the origin if the line pair, $xy - 3x + 2y - 6 = 0$ is orthogonal to it. If this circle is orthogonal to the circle $x^2 + y^2 - kx + 2ky - 8 = 0$ then find the value of k.

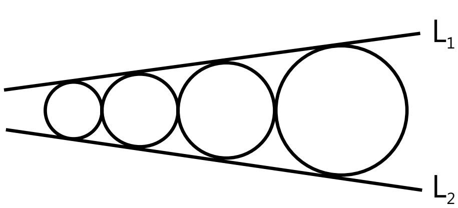
Q.13 Find the equation of the circle which cuts the circle $x^2 + y^2 - 14x - 8y + 64 = 0$ and the coordinate axes orthogonally.

Q.14 Show that the locus of the centres of a circle which cuts two given circles orthogonally is a straight line & hence deduce the locus of the centres of the circles which cut the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ & $x^2 + y^2 - 5x + 4y + 2 = 0$ orthogonally. Intercept the locus.

Q.15 Find the equation of a circle which touches the line x + y = 5 at the point (–2, 7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally.

Q.16 Find the equation of the circle passing through the point (–6, 0) if the power of the point (1, 1) w.r.t. the circle is 5 and it cuts the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ orthogonally.

Q.17 As shown in the figure, the five circles are tangent to one another consecutively and to the lines L_1 and L_2. If the radius of the largest circle is 18 and that of the smaller one is 8, then find the radius of the middle circle.



Q.18 Find the equation of a circle which touches the line $7x^2 - 18xy + 7y^2 = 0$ and the circle $x^2 + y^2 - 8x - 8y = 0$ and is contained in the given circle.

Q.19 Consider two circle C_1 of radius 'a' and C_2 of radius 'b' (b > a) both lying in the first quadrant and touching the coordinate axes. In each of the conditions listed in column-I, the ratio of b/a is

Column I	**Column II**
(A) C_1 and C_2 touch each other	(p) $2 + \sqrt{2}$
(B) C_1 and C_2 are orthogonal	(q) 3
(C) C_1 and C_2 intersect so that the common chord is longest	(r) $2 + \sqrt{3}$
(D) C_2 passes through the centre of C_1	(s) $3 + 2\sqrt{2}$
	(t) $3 - 2\sqrt{2}$

Q.20 A circle with centre in the first quadrant is tangent to y = x + 10, y = x – 6, and the y-axis. Let (h, k) be the centre of the circle. If the value of $(h + k) = a + b\sqrt{a}$ where $\sqrt{a}$ is a surd, find the value of a + b.

Q.21 Circles C_1 and C_2 are externally tangent and they are both internally tangent to the circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively and the centres of the three circles are collinear. A chord of C_3 is also a common internal tangent of C_1 and C_2. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$ where m, n and p are positive integers, m and p are relatively prime and n is not divisible by the square of any prime, find the value of (m + n + p).

Q.22 Find the equation of the circle passing through the three points (4, 7), (5, 6) and (1, 8). Also find the coordinates of the point of intersection of the tangents to the circle at the points where it is cut by the straight line 5x + y + 17 = 0.

Q.23 The line 2x – 3y + 1 = 0 is tangent to a circle S = 0 at (1, 1). If the radius of the circle is $\sqrt{13}$. Find the equation of the circle S.

Q.24 Find the equation of the circle which passes through the point (1, 1) & which touches the circle

$x^2 + y^2 + 4x - 6y - 3 = 0$ at the point (2, 3) on it.

Exercise 2

Single Correct Choice Type

Q.1 B and C are fixed points having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation :

(A) $x^2 + y^2 = 1$ (B) $x^2 + y^2 = 2$

(C) $9(x^2 + y^2) = 1$ (D) $9(x^2 + y^2) = 4$

Q.2 Number of points in which the graphs of $|y| = x + 1$ and $(x - 1)^2 + y^2 = 4$ intersect, is

(A) 1 (B) 2 (C) 3 (D) 4

Q.3 $y - 1 = m_1(x - 3)$ and $y - 3 = m_2(x - 1)$ are two family of straight lines, at right angles to each other. The locus of their point of intersection is

(A) $x^2 + y^2 - 2x - 6y + 10 = 0$

(B) $x^2 + y^2 - 4x - 4y + 6 = 0$

(C) $x^2 + y^2 - 2x - 6y + 6 = 0$

(D) $x^2 + y^2 - 4x - 4y - 6 = 0$

Q.4 The points (x_1, y_1), (x_2, y_2), (x_1, y_2) and (x_2, y_1) are always:

(A) Collinear (B) Concyclic

(C) Vertices of a square (D) Vertices of a rhombus

Q.5 Consider 3 non-collinear points A, B, C with coordinates (0, 6), (5, 5) and (–1, 1) respectively. Equation of a line tangent to the circle circumscribing the triangle ABC and passing through the origin is

(A) $2x - 3y = 0$ (B) $3x + 2y = 0$

(C) $3x - 2y = 0$ (D) $2x + 3y = 0$

Q.6 A (1, 0) and B(0, 1) and two fixed points on the circle $x^2 + y^2 = 1$. C is a variable point on this circle. As C moves, the locus of the orthocenter of the triangle ABC is

(A) $x^2 + y^2 - 2x - 2y + 1 = 0$

(B) $x^2 + y^2 - x - y = 0$

(C) $x^2 + y^2 = 4$

(D) $x^2 + y^2 + 2x - 2y + 1 = 0$

Q.7 A straight line with slope 2 and y - intercept 5 touches the circle, $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are

(A) (–6, 11) (B) (–9, –13)

(C) (–10, –15) (D) (–6, –7)

Q.8 A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is

(A) $8\sqrt{3}$ sq. units (B) $4\sqrt{3}$ sq. units

(C) $16\sqrt{3}$ sq. units (D) None of these

Q.9 From (3, 4) chords are drawn to the circle $x^2 + y^2 - 4x = 0$. The locus of the mid points of the chords is:

(A) $x^2 + y^2 - 5x - 4y + 6 = 0$

(B) $x^2 + y^2 + 5x - 4y + 6 = 0$

(C) $x^2 + y^2 - 5x + 4y + 6 = 0$

(D) $x^2 + y^2 - 5x - 4y - 6 = 0$

Q.10 The line joining (5, 5) to ($10 \cos \theta$, $10 \sin \theta$) is divided internally in the ratio 2 : 3 at P. If θ varies then the locus of P is :

(A) A pair of straight lines

(B) A circle

(C) A straight line

(D) A second degree curve which is not a circle

Q.11 The normal at the point (3, 4) on a circle cuts the circle at the point (–1, –2). Then the equation of the circle is:

(A) $x^2 + y^2 + 2x - 2y - 13 = 0$

(B) $x^2 + y^2 - 2x - 2y - 11 = 0$

(C) $x^2 + y^2 - 2x + 2y + 12 = 0$

(D) $x^2 + y^2 - 2x - 2y + 14 = 0$

Q.12 The shortest distance from the line $3x + 4y = 25$ to the circle $x^2 + y^2 = 6x - 8y$ is equal to

(A) $\frac{7}{5}$ (B) $\frac{9}{5}$ (C) $\frac{11}{5}$ (D) $\frac{32}{5}$

Q.13 The equation of a line inclined at an angle $\frac{\pi}{4}$ to the axis X, such that the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal lengths on it, is

(A) $2x - 2y - 3 = 0$ (B) $2x - 2y + 3 = 0$

(C) $x - y + 6 = 0$ (D) $x - y - 6 = 0$

Q.14 The locus of the midpoint of a line segment that is drawn from a given external point P to a given circle with centre O (where O is origin) and radius r, is

(A) A straight line perpendicular to PO

(B) A circle with centre P and radius r

(C) A circle with centre P and radius 2r

(D) A circle with centre at the midpoint PO and radius $\frac{r}{2}$

Multiple Correct Choice Type

Q.15 Locus of the intersection of the two straight lines passing through (1, 0) and (–1, 0) respectively and including an angle of $45°$ can be a circle with

(A) Centre (1, 0) and radius $\sqrt{2}$.

(B) Centre (1, 0) and radius 2.

(C) Centre (0, 1) and radius $\sqrt{2}$.

(D) Centre (0, –1) and radius $\sqrt{2}$.

Q.16 Consider the circles

$S_1 : x^2 + y^2 + 2x + 4y + 1 = 0$

$S_2 : x^2 + y^2 - 4x + 3 = 0$

$S_3 : x^2 + y^2 + 6y + 5 = 0$

Which of this following statement are correct?

(A) Radical centre of S_1, S_2 and S_3 lies in 1st quadrant.

(B) Radical centre of S_1, S_2 and S_3 lies in 4st quadrant.

(C) Radical centre of S_1, S_2 and S_3 orthogonally is 1.

(D) Circle orthogonal to S_1, S_2 and S_3 has its x and y intercept equal to zero.

Q.17 Consider the circles

$C_1 : x^2 + y^2 - 4x + 6y + 8 = 0$

$C_2 : x^2 + y^2 - 10x - 6y + 14 = 0$

Which of the following statement (s) hold good in respect of C_1 and C_2?

(A) C_1 and C_2 are orthogonal.

(B) C_1 and C_2 touch each other.

(C) Radical axis between C_1 and C_2 is also one of their common tangent.

(D) Middle point of the line joining the centres of C_1 and C_2 lies on their radical axis.

Q.18 A circle passes through the points (–1 , 1), (0, 6) and (5, 5). The point (s) on this circle, the tangent (s) at which is/are parallel to the straight line joining the origin to its centre is/are:

(A) (1, –5) (B) (5, 1) (C) (–5, –1) (D) (–1, 5)

Q.19 The circles $x^2 + y^2 + 2x + 4y - 20 = 0$ and $x^2 + y^2 + 6x - 8y + 10 = 0$

(A) Are such that the number of common tangents on them is 2

(B) Are not orthogonal

(C) Are such that the length of their common tangent is $5\left(\frac{12}{5}\right)^{\frac{1}{4}}$

(D) Are such that the length of their common chord is $5\,\frac{\sqrt{3}}{2}$.

Q.20 Three distinct lines are drawn in a plane. Suppose there exist exactly n circles in the plane tangent to all the three lines, then the possible values of n is/are

(A) 0 (B) 1 (C) 2 (D) 4

Q.21 The equation of a circle C_1 is $x^2+y^2+14x-4y+28=0$. The locus of the point of intersection of orthogonal tangents to C_1 is the curve C_2 and the locus of the point of intersection of perpendicular tangents to C_2 is the curve C_3 then the statement (s) which hold good?

(A) C_3 is a circle

(B) Area enclosed by C_3 is 100π sq. unit

(C) Area of C_2 is $\sqrt{2}$ times the area of C_1.

(D) C_2 and C_3 are concentric circles.

Q.22 The circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 4x + 4y - 1 = 0$

(A) Touch internally

(B) Touch externally

(C) Have $3x + 4y - 1 = 0$ as the common tangent at the point of contact.

(D) have $3x + 4y + 1 = 0$ as the common tangent at the point of contact.

Q.23 Which of the following is/are True? The circles $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 + 6x + 6y + 9 = 0$ are such that

(A) They do not intersect.

(B) They touch each other.

(C) Their direct common tangents are parallel.

(D) Their trannsverse common tangents are perpendicular.

Q.24 Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ intersect each other orthogonally then the value of p is

(A) 1 (B) 2 (C) 3 (D) 5

Q.25 Which of the following statements is/are incorrect?

(A) Two circles always have a unique common normal.

(B) Radical axis is always perpendicular bisector to the line joining the centres of two circles.

(C) Radical axis is nearer to the centre of circle of smaller radius.

(D) Two circles always have a radical axis.

Assertion Reasoning Type

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I.

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I.

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

Q.26 Consider the lines L : $(k + 7)x - (k - 1)y - 4(k - 5)=0$ where k is a parameter and the circle

C : $x^2 + y^2 + 4x + 12y - 60 = 0$

Statement-I: Every member of L intersects the circle 'C' at an angle of $90°$

Statement-II: Every member of L tangent to the circle C.

Q.27 Statement-I: Angle between the tangents drawn from the point P(13, 6) to the circle S : $x^2 + y^2 - 6x + 8y - 75 = 0$ is $90°$.

Statement-II: Point P lies on the director circle of S.

Q.28 Statement-I: From the point (1, 5) as its centre, only one circle can be drawn touching the circle $x^2 + y^2 - 2x = 7$.

Statement-II: Point (1, 5) lies outside the circle $x^2 + y^2 - 2x = 7$.

Q.29 Statement-I: Let C_1 (0, 0) and C_2(2, 2) be centres of two circle and L : $x + y - 2 = 0$ is their common chord. If length of common chord is equal to $\sqrt{2}$, then both circles intersect orthogonally.

Statement-II: Two circles will be orthogonal if their centres are mirror images of each other in their common chord and distance between centres is equal to length of common chord.

Q.30 Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, x_3)$ are the vertices of a triangle ABC.

Statement-I : If angel C is obtuse then the quantity $(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2)$ is negative.

Statement-II: Diameter of a circle subtends obtuse angle at any point lying inside the semicircle.

Q.31 Let C be a circle with centre 'O' and HK is the chord of contact of pair of the tangents from point A. OA intersects the circle C at P and Q and B is the midpoint of HK, then

Statement-I: AB is the harmonic mean of AP and AQ.

Statement-II: AK is the Geometric mean of AB and AO and OA is the arithmetic mean of AP and AQ.

Comprehension Type

Paragraph for questions 32 to 34

Let A, B, C be three sets of real numbers (x, y) defined as

A : $\{(x, y): y \geq 1\}$

B : $\{(x, y): x^2 + y^2 - 4x - 2y - 4 = 0\}$

C : $\{(x, y): x + y = \sqrt{2}\}$

Q.32 Number of elements in the $A \cap B \cap C$ is

(A) 0 (B) 1 (C) 2 (D) infinite

Q.33 $(x + 1)^2 + (y - 1)^2 + (x - 5)^2 + (y - 1)^2$ has the value equal to

(A) 16 (B) 25 (C) 36 (D) 49

Q.34 If the locus of the point of intersection of the pair of perpendicular tangents to the cirlc B is the curve S then the area enclosed between B and S is

(A) 6π (B) 8π (C) 9π (D) 18π

Paragraph for questions 35 to 36

Consider a circle $x^2 + y^2 = 4$ and a point P(4, 2). θ denotes the angle enclosed by the tangents from P on the circle and A, B are the points of contact of the tangents from P on the circle.

Q.35 The value of θ lies in the interval

(A) $(0, 15°)$ (B) $(15°, 30°)$

(C) $(30°, 45°)$ (D) $(45°, 60°)$

Q.36 The intercept made by a tangent on the x-axis is

(A) $\frac{9}{4}$ (B) $\frac{10}{4}$ (C) $\frac{11}{4}$ (D) $\frac{12}{4}$

Paragraph for questions 37 to 39

Consider the circle S : $x^2 + y^2 - 4x - 1 = 0$ and the line L : $y = 3x - 1$. If the line L cuts the circle at A and B then

Q.37 Length of the chord AB equal

(A) $2\sqrt{5}$ (B) $\sqrt{5}$ (C) $5\sqrt{2}$ (C) $\sqrt{10}$

Q.38 The angle subtended by the chord AB in the minor arc of S is

(A) $\frac{3\pi}{4}$ (B) $\frac{5\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{\pi}{4}$

Q.39 Acute angel between the line L and the circle S is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Previous Years' Questions

Q.1 Tangents drawn from the point P (1, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the point A and B. The equation of the circumcircle of the triangle PAB is ***(2009)***

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$

(B) $x^2 + y^2 - 4x - 10y + 19 = 0$

(C) $x^2 + y^2 - 2x + 6y - 29 = 0$

(D) $x^2 + y^2 - 6x - 4y + 19 = 0$

Q.2 Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and AB = 2CD. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, its radius is ***(2007)***

(A) 3 (B) 2 (C) $\frac{3}{2}$ (D) 1

Q.3 The locus of the centre of circle which touches $(y - 1)^2 + x^2 = 1$ externally and also touches x axis, is ***(2005)***

(A) $\{x^2 = 4y\ ,\ y \ge 0\} \cup \{(0, y),\ y < 0\}$

(B) $x^2 = y$

(C) $y = 4x^2$

(D) $y^2 = 4x \cup (0, y),\ y \in R$

Q.4 Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r If PS and RQ intersect at a point X on the circumference of the circle, then 2r equals ***(2001)***

(A) $\sqrt{PQ \cdot RS}$ (B) $\frac{PQ + RS}{2}$

(C) $\frac{2PQ \cdot RS}{PQ + RS}$ (D) $\sqrt{\frac{PQ^2 + RS^2}{2}}$

Q.5 Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the locus of centroid of the triangle PAB as P moves on the circle is ***(2001)***

(A) A parabola (B) A circle

(C) An ellipse (D) A pair of straight lines

Q.6 If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \ne 0$) are bisected by the x-axis, then ***(1999)***

(A) $p^2 = q^2$ (B) $p^2 = 8q^2$

(C) $p^2 < 8q^2$ (D) $p^2 > 8q^2$

Q.7 Consider

$L_1 : 2x + 3y + p - 3 = 0$

$L_2 : 2x + 3y + p + 3 = 0$

where p is a real number and

C : $x^2 + y^2 - 6x + 10y + 30 = 0$ ***(2008)***

Statement-I: If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C. and

Statement-II: If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

Paragraph 1: Let ABCD be a square of side length 2 unit. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of square ABCD. L is the line through A. ***(2006)***

Q.8 If P is a point of C_1 and Q is a point on C_2, then $\frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ is equal to

(A) 0.75 (B) 1.25 (C) 1 (D) 0.5

Q.9 A circle touches the line L and the circle C_1 externally such that both the circle are on the same side of the line, then the locus of centre of the circle is

(A) Ellipse (B) Hyperbola

(C) Parabola (D) Parts of straight line

Q.10 A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1. then area of $\Delta T_1T_2T_3$ is

(A) $\frac{1}{2}$ sq unit (B) $\frac{2}{3}$ sq unit

(C) 1 sq unit (D) 2 sq unit

Paragraph 2: A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ. ***(2008)***

Q.11 The equation of circle C is

(A) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$

(B) $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$

(C) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

(D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Q.12 Point E and F are given by

(A) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\sqrt{3}, 0\right)$ (B) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\sqrt{3}, 0\right)$

(C) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Q.13 Equations of the sides QR, RP are

(A) $y = \frac{2}{\sqrt{3}}x + 1,\ y = -\frac{2}{\sqrt{3}}x - 1$

(B) $y = \frac{1}{\sqrt{3}}x,\ y = 0$

(C) $y = \frac{\sqrt{3}}{2}x + 1,\ y = -\frac{\sqrt{3}}{2}x - 1$

(D) $y = \sqrt{3}x,\ y = 0$

Q.14 Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA. ***(2001)***

Q.15 Let T_1, T_2 and be two tangents drawn from (–2, 0) onto the circle C : $x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles when taken two at a time. ***(1999)***

Q.16 Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where $k > 0$, then the value of $[k]$ is

[**Note :** $[k]$ denotes the largest integer less than or equal to k] ***(2010)***

Q.17 The circle passing through the point $(-1,0)$ and touching the $y-$axis at $(0,2)$ also passes through the point ***(2011)***

(A) $\left(-\frac{3}{2}, 0\right)$ (B) $\left(-\frac{5}{2}, 2\right)$

(C) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (D) $(-4, 0)$

Paragraph 3: A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P\left(\sqrt{3}, 1\right)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2 + y^2 = 1$.

Q. 18 A common tangent of the two circles is ***(2012)***

(A) $x = 4$ (B) $y = 2$

(C) $x + \sqrt{3}y = 3$ (D) $x + 2\sqrt{2}y = 6$

Q.19 A possible equation of L is ***(2012)***

(A) $x - \sqrt{3}y = 1$ (B) $x + \sqrt{3}y = 1$

(C) $x - \sqrt{3}y = -1$ (D) $x + \sqrt{3}y = 5$

Q.20 Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is ***(2012)***

Q.21 The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is ***(2012)***

(A) $20(x^2 + y^2) - 36x + 45y = 0$

(B) $20(x^2 + y^2) + 36x$ ñ $45y = 0$

(C) $36(x^2 + y^2) - 20x + 45y = 0$

(D) $36(x^2 + y^2) + 20x$ ñ $45y = 0$

Q.22 Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are) ***(2013)***

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$

(B) $x^2 + y^2 - 6x + 7y + 9 = 0$

(C) $x^2 + y^2 - 6x - 8y + 9 = 0$

(D) $x^2 + y^2 - 6x - 7y + 9 = 0$

Q.23 The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral PQRS is ***(2014)***

(A) 3 (B) 6 (C) 9 (D) 15

Q.24 A circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then ***(2014)***

(A) Radius of S is 8 (B) Radius of S is 7

(C) Centre of S is (-7, 1) (D) Centre of S is (-8, 1)

Q.25 Let

$$f(x) = \lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}},$$

for all x > 0. Then ***(2016)***

(A) $f\left(\frac{1}{2}\right) \ge f(1)$ (B) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$

(C) $f(2) \le 0$ (D) $\frac{f'(3)}{f(3)} \ge \frac{f'(2)}{f(2)}$

Q.26 The circle $C_1 : x^2 + y^2 = 3$, with centre O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3 respectively. If Q_2 and Q_3 lie on the y-axis, then ***(2016)***

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) Area of the triangle OR_2R_3 is $6\sqrt{2}$

(D) Area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Q.27 Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1, 0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s) ***(2016)***

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$

(B) $\left(\frac{1}{4}, \frac{1}{2}\right)$

(C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$

(D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.12	Q.18	Q.21
Q.23	Q.29	

Exercise 2

Q.3	Q.7	Q.14
Q.15	Q.20	

Previous Years' Questions

Q.1	Q.3	Q.5
Q.8	Q.11	Q.13

JEE Advanced/Boards

Exercise 1

Q.5	Q.9	Q.14	
Q.17	Q.19	Q.21	Q.24

Exercise 2

Q.2	Q.4	Q.9
Q.13	Q.16	Q.21
Q.22	Q.25	Q.27
Q.22	Q.25	Q.26
Q.29	Q.32	

Previous Years' Questions

Q.1	Q.3	Q.6
Q.7	Q.9	Q.13

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $x^2 + y^2 + 3x + 12y + 2 = 0$

Q.3 $(1, -1)$, $\sqrt{13}$, $x^2 + y^2 - 2x + 2y - 11 = 0$

Q.4 $3x - 4y = 7$, $4x + 3y = 0$

Q.6 $13x + 9y = 77$, $3x - y - 27 = 0$

Q.7 $2x^2 + 2y^2 + 16x - 8y - 41 = 0$

Q.8 $23x^2 + 23y^2 - 156x + 38y + 168 = 0$

Q.9 $y = x$

Q.10 $x = 3$

Q.11 $\dfrac{\sqrt{109}}{2}$

Q.12 $x = 3$ and $y = \pm 3$.

Q.14 $x^2 + y^2 + 2ax + 2py - (b^2 + q^2) = 0$; $\sqrt{a^2 + b^2 + p^2 + q^2}$

Q.15 $x^2 + y^2 + 18x - 2y + 32 = 0$

Q.16 32 sq. units

Q.17 $x^2 + y^2 - 12x - 10y + 52 = 0$

Q.20 $4\sqrt{2}$

Q.21 $\cos^{-1} \dfrac{1}{2\sqrt{2}}$

Q.22 $x^2 + y^2 - 2x - 6y - 8 = 0$

Q.23 8 sq. units.

Q.26 $(2, -1)$; 2

Q.28 $4(x^2 + y^2) + 2y - 29 = 0$

Q.29 $x^2+y^2+8x-6y+9=0$ **Q.30** $x^2+y^2-x=0$ **Q.31** $2a^2=b^2$

Exercise 2

Single Correct Choice Type

Q.1 D **Q.2** B **Q.3** A **Q.4** B **Q.5** A **Q.6** D

Q.7 D **Q.8** B **Q.9** C **Q.10** C **Q.11** A **Q.12** D

Q.13 C **Q.14** A **Q.15** B **Q. 16** C **Q.17** A **Q.18** D

Q.19 C **Q.20** D **Q.21** A

Previous Years' Questions

Q.1 D **Q.2** B **Q.3** C **Q.4** A **Q.5** C **Q.6** A

Q.7 C **Q.8** B **Q.10** $2x^2+2y^2-10x-5y+1=0$ **Q.11** (a,b) and (0,0)

Q.12 $4x^2+25y^2=4x^2y^2$ **Q.14** A **Q.15** B **Q.16** D **Q.17** D

Q.18 B **Q.19** C

JEE Advanced/Boards

Exercise 1

Q.1 (i) $x^2+y^2-4x-3y=0$, (ii) $x^2+y^2=24$, (iii) $4x+3y=24$ **Q.2** 32

Q.3 625 **Q.4** 4 **Q.6** $\left(2,\frac{23}{3}\right)$

Q.7 $x^2+y^2-3x-3y+4=0$ **Q.8** (1, 0)& $\left(\frac{1}{2},\frac{1}{2}\right)$; $r=\frac{1}{2\sqrt{2}}$ **Q.9** $4x^2+4y^2+6x+10y-1=0$

Q.10 $x^2+y^2+16x+14y-12=0$ **Q.11** (–4, 4); $\left(-\frac{1}{2},\frac{1}{2}\right)$ **Q.12** $x^2+y^2+4x-6y=0$; k= 1;

Q.13 $x^2+y^2=64$ **Q.14** $9x-10y+7=0$; radical axis **Q.15** $x^2+y^2+7x-11y+38=0$

Q.16 $x^2+y^2+6x-3y=0$ **Q.17** 12 **Q.18** $x^2+y^2-12x-12y+64=0$

Q.19 (A) S; (B) R; (C) Q ; (D) P **Q.20** 10 **Q.21** 19

Q.22 (–4, 2), $x^2+y^2-2x-6y-15=0$

Q.23 $x^2+y^2-6x+4y=0$ OR $x^2+y^2+2x-8y+4=0$ **Q.24** $x^2+y^2+x-6y+3=0$

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** B	**Q.4** B	**Q.5** D	**Q.6** A
Q.7 D	**Q.8** A	**Q.9** A	**Q.10** B	**Q.11** B	**Q.12** A
Q.13 A	**Q.14** D				

Multiple Correct Choice Type

Q.15 C, D	**Q.16** B, C, D	**Q.17** B, C	**Q.18** B, D	**Q.19** A, C, D	**Q.20** A, C, D
Q.21 A, B, D	**Q.22** B, C	**Q.23** A, C, D	**Q.24** B, C	**Q.25** A, B, D	

Assertion Reaosing Type

Q.26 C	**Q.27** A	**Q.28** D	**Q.29** A	**Q.30** A	**Q.31** A

Comprehension Type

Paragraph 1:	**Q.32** B	**Q.33** C	**Q.34** C
Paragraph 2:	**Q.35** D	**Q.36** B	
Paragraph 3:	**Q.37** D	**Q.38** A	**Q.39** C

Previous Years' Questions

Q.1 B	**Q.2** B	**Q.3** A	**Q.4** A	**Q.5** B	**Q.6** D
Q.7 C	**Q.8** A	**Q.9** C	**Q.10** C	**Q.11** D	**Q.12** A
Q.13 D	**Q.14** $3(3+\sqrt{10})$	**Q.15** $\left(X+\frac{4}{3}\right)^2 + y = \left(\frac{1}{3}\right)^2; y = \pm\frac{5}{\sqrt{39}}\left(x+\frac{4}{5}\right)$			**Q.16** 3
Q.17 D	**Q.18** D	**Q.19** A	**Q.20** 4	**Q.21** A	**Q.22** C
Q.23 D	**Q.24** B C	**Q.25** A C D	**Q.26** C	**Q.27** A C	

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\therefore$ Centre lies on $2x - y - 3 = 0$

$\therefore$ Let the centre be $C \equiv (h, 2h - 3)$

It also passes through $A \equiv (3, -2)$ and $B \equiv (-2, 0)$

$\therefore$ AC = BC

$\Rightarrow (h - 3)^2 + (2h - 1)^2 = (h + 2)^2 + (2h - 3)^2$

$\Rightarrow -6h + 9 - 4h + 1 = 4h + 4 - 12h + 9 - 2h = 3$

$\therefore h = -\dfrac{3}{2}$ $\quad \therefore C = \left(\dfrac{-3}{2}, -6\right)$

$\therefore$ Equation of the circle is

$(x - h)^2 + (y - k)^2 = R^2$

$\Rightarrow \left(x + \dfrac{3}{2}\right)^2 + (y + 6)^2 = \left(\dfrac{-3}{2}, -3\right)^2 + (-6 + 2)^2$

$\Rightarrow x^2 + 3x + \dfrac{9}{4} + y^2 + 12y + 36 = \dfrac{81}{4} + 16$

$x^2 + y^2 + 3x + 12y + 2 = 0$

Sol 2: We can see that (0, 0), (1, 1) & (6, –4) form a right angled triangle with (0, 0) & (6, –4) as diameter

Equation of circle is $(x - 0)(x - 6) + y(y + 4) = 0$

$\Rightarrow C = x(x - 6) + y(y + 4) = 0$

We can see that (5, –5) satisfies this equation

$\therefore$ 4 points are concyclic

Sol3: $A \equiv (-1, 2)$, and $B \equiv (3, -4)$

Equation of the circle is $(x + 1)(x - 3) + (y - 2)(y + 4)$

$\Rightarrow x^2 + y^2 - 2x + 2y - 11 = 0$

$\therefore C = (-g, -f) = (1, -1)$

Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{1 + 1 + 11} = \sqrt{13}$

Sol 4: Given equation of circle is $x^2 + y^2 = 25$ $P \equiv (-3, -4)$

$\therefore$ Slope of normal $OP = \dfrac{0 + 4}{0 + 3} = \dfrac{4}{3}$

$\therefore$ Equation of normal is $(y + 4) = \dfrac{4}{3}(x + 3)$

$\therefore$ Slope of tangent at P is $\dfrac{-1}{4/3} = \dfrac{-3}{4}$

$\therefore$ Equation of tangent is $(y + 4) = \dfrac{-3}{4}(x + 3)$

$\Rightarrow 4y + 16 = -3x - 9$

$\Rightarrow 3x + 4y + 25 = 0$

Sol 5: Equation of tangent at (1, –2) is

$x\,x1 + y\,y1 = a^2$

$x - 2y = 5$

$\therefore y = \dfrac{1}{2}x - \dfrac{5}{2}$

Equation of C_2 is $(x - 4)^2 + (x + 3)^2 = (\sqrt{5})^2$

Now the tangent will touch C_2 If $c^2 = r^2(1 + m^2)$

$c^2 = \left(\dfrac{5}{2}\right)^2$

$r^2(1 + m^2) = 5 \times \left(1 + \dfrac{1}{4}\right) = \left(\dfrac{5}{2}\right)^2$

$\therefore$ The given line is tangent to C_2

Sol 6: Equation of circle is

$C \equiv (x - 1)^2 + (y + 4)^2 = (2\sqrt{10})^2$

Shifting origin to (1, –4)

$\therefore C' \equiv X^2 + Y^2 = (2\sqrt{10})^2$ & $P = (7, 1)$

$\therefore Y - 1 = m(X - 7)$

$\therefore Y = mX + (1 - 7m)$

$\therefore c^2 = a^2(1 + m^2)$

$(1 - 7m)^2 = 40(1 + m^2)$

$\Rightarrow 9m^2 - 14m - 39 = 0$

$\Rightarrow 9m^2 - 27m + 13m - 39 = 0$

$m = 3$ or $m = \dfrac{-13}{9}$

Since slope remains same in both system

$\therefore$ Equation of lines in old co-ordinates are

$(y + 3) = 3(x - 8)$ $\quad$ & $(y + 3) = \dfrac{-13}{9}(x - 8)$

Or $3x - y - 27 = 0$ and $13x + 9y = 77$

Sol 7: Centre = (– 4, 2)

Tangent is x – y = 3

$\therefore$ Radius = $\left|\frac{-4-2-3}{\sqrt{2}}\right| = \left(\frac{9}{\sqrt{2}}\right)$

$\therefore C \equiv (x+4)^2 + (y-2)^2 = \frac{81}{2}$

Sol 8: Using the concept of family of circles, let the equation of circle be

$\left(x^2+y^2-8x-2y+7\right)+\lambda\left(x^2+y^2-4x+10y+8\right)=0$

As (3, -3) lies on it

$\therefore \left(9+9-24+6+7\right)+\lambda\left(9+9-12-30+8\right)=0$

$\Rightarrow 7-16\lambda=0$

$\Rightarrow \lambda=\frac{7}{16}$

$\therefore$ Equation of the circle is

$\left(x^2+y^2-8x-2y+7\right)+\frac{7}{16}\left(x^2+y^2-4x+10y+8\right) = 0$

or $23x^2+23y^2-156x+38y+168=0$

Sol 9: $C \equiv x^2 + y^2 - 4x = 0$

Centre = (2, 0)

Slope of line perpendicular to chord $=\frac{1-0}{1-2}=-1$

$\therefore$ Slope of chord = 1

$\Rightarrow$ y – 1 = 1 (x – 1)

$\therefore$ y = x is the equation of chord

Alternative

Equation of a chord bisected at a given point is $T=S_1$

$\therefore xx_1+yy_1-2\left(x+x_1\right)=x_1^2+y_1^2-4x_1$

Or, $x+y-2x-2=1+1-4$

Or, x – y = 0

Sol 10: Equation of chord of contact

$xx_1 + yy_1 + g(x + x_1) + f(g+g_1) + c = 0$

$\Rightarrow 6x - 2(x + 6) = 0$; x = 3

Sol 11: Length of tangent from a point

$= \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+C}$

$= \sqrt{\frac{4(3)^2+4(2)^2+4\times3+16\times2+13}{4}} = \sqrt{\frac{109}{4}} = \frac{\sqrt{109}}{2}$

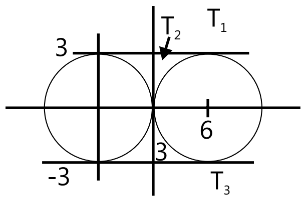
Sol 12: $C_1 \equiv x^2 + y^2 = 9$

Centre = (0, 0 & R_1 = 3)

$C_2 \equiv x^2 + y^2 - 12x + 27 = 0$

Centre = (6, 0 & R_2 = 3)

$\therefore$ The circles touch each other externally



$\therefore$ The equation of tangents are

y = 3, x = 3 & y = –3 (from figure itself)

Sol 13: Family of circles passing through two points is

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + \lambda L = 0$

$\therefore x(x-1) + y^2 + \lambda y = 0$

$\therefore x^2 + y^2 - x + \lambda y = 0$

Centre = $\left(\frac{1}{2},\frac{-\lambda}{2}\right)$

Now since the circle touches internally [$\because$ (0, 0), & (1, 0) lie inside the circle]

$\therefore r_1 - r_2$ = distance between their centres

$\therefore 3-\sqrt{\frac{1}{4}+\frac{\lambda^2}{4}} = \sqrt{\frac{1}{4}+\frac{\lambda^2}{4}}$

$\therefore 9 = 4\left(\frac{1+\lambda^2}{4}\right)$

$\therefore \lambda = \pm2\sqrt{2}$ $\quad\therefore$ Centre = $\left(\frac{1}{2},\pm\sqrt{2}\right)$

Sol 14: Let the coordinates of diameter be (h_1, k_1) & (h_2, k_2)

$\therefore$ Equation of circle is

$(x-h_1)(x-h_2)+(y-k_1)(y-k_2) = 0$

$\Rightarrow x^2 - y^2 - (h_1 + h_2)x - (k_1 - k_2)y + (h_1h_2+k_1k_2)=0$

$\Rightarrow x^2 + y^2 - (-2a)x - (-2p)y + (-b^2 - q^2) = 0$

$\Rightarrow x^2 + y^2 + 2ax + 2py - (b^2 + q^2) = 0$

$\therefore R = \sqrt{a^2+p^2+b^2+q^2}$

Sol 15: Given equation of line is x + y = 2 ...(i)

On solving (i), with $x^2+y^2=2$, we get

$x^2+(2-x)^2=2$

$\Rightarrow x^2+4-4x+x^2=2$

$\Rightarrow 2x^2-4x+2=0$

$\Rightarrow (x-1)^2=0 \Rightarrow x=1$ This means the line represented by (i) and the circle intersects only at (1, 1)

Similarly, on solving x + y = 2 and

$x^2+y^2+3x+3y-8=0$, we get

$2x^2-4x+4+3(2)-8=0$, we get

$\Rightarrow 2x^2-4x+2=0$

$\Rightarrow 2(x-1)^2=0$

$\Rightarrow x=1$

Hence, the line intersects only at one point (1, 1)

Hence, proved.

Sol 16: Let $C \equiv (h, k)$ be the center of the circle

$\therefore 4k = h + 7$

$\therefore$ AC = BC

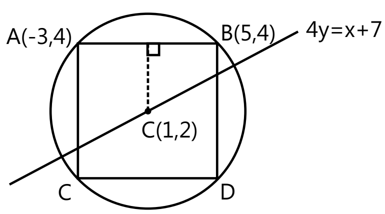
$\Rightarrow (4k-7+3)^2+(k-4)^2=(4k-12)^2+(k-4)^2$

$\Rightarrow (4k-4)^2=(4k-12^2)$

$\Rightarrow K=2$

$\therefore C \equiv (1, 2)$

Now



Equation of chord AB is y – 4 = 0

$\therefore$ Perpendicular distance of center from chord AB is

$\left|\frac{2-4}{1}\right| = 2$

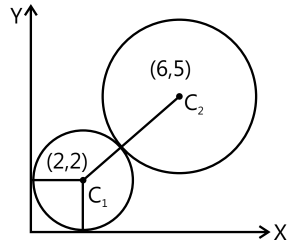
$\therefore$ AB = 8 and BC = 2PQ = 4

$\therefore$ Area of rectangle = 8 × 4 = 32

Sol 17: Radius of C_1 = 2

$\therefore$ Centre = (2, 2)

As circle with center (6,5) touches it externally



$\therefore C_1C_2=r_1+r_2$

$\Rightarrow \sqrt{(6-2)^2+(5-2)^2} = 2 + r$

$\Rightarrow r^2 + 4r + 4 = 16 + 9$

$\Rightarrow r^2 + 4r - 21 = 0$

$\Rightarrow r^2 + 7r - 3r - 21 = 0$

$\therefore r = 3$ ($\because$ r cannot be negative)

$\therefore$ Equation of C_2 is $(x-6)^2+(y-5)^2=9$

Sol 18: Let the equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

Let $\left(m, \frac{1}{m}\right)$ be point on the circle.

On substitution we get

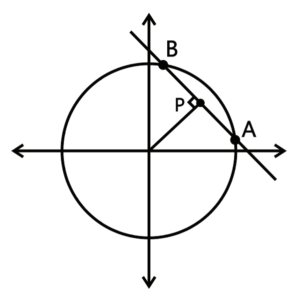
If $m^4 + 2gm^3 + 2fm + cm^2 + 1 = 0$

m_1, m_2, m_3, m_4 are roots of this equation

then, $m_1m_2m_3m_4 = 1$

Sol 19: Equation of line

$\Rightarrow$ x cos x + y sin x – p = 0



Now family of circle passing through the intersection of the circle & line is

$x^2 + y^2 - a^2 + \lambda$ (x cos x + y sin x – p) = 0

$\therefore$ Radius of circle = $AM=\sqrt{a^2-p^2}$

$\Rightarrow \sqrt{\left(\frac{\lambda\cos x}{2}\right)^2+\left(\frac{\lambda\sin x}{2}\right)^2+a^2+\lambda p} = \sqrt{a^2-p^2}$

$\Rightarrow \frac{\lambda^2}{4} + \lambda p + p^2 = 0$

$(\lambda + 2p)^2 = 0 \Rightarrow \lambda = -2p$

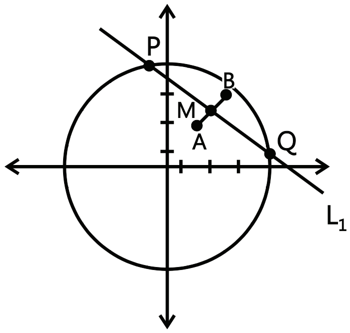
$\therefore S \equiv x^2 + y^2 - 2px\cos x$

$- 2py\sin x + 2P^2 - a^2 = 0$

Sol 20: Slope of AB = 1

$\therefore$ Slope of $L_1 \times$ slope of $AB = -1$

$\Rightarrow$ Slope of $L_1 = -1$



And mid-point of AB, $M \equiv \left(\frac{3}{2}, \frac{5}{2}\right)$

$\therefore$ Equation of line L_1 is

$\left(y - \frac{5}{2}\right) = -1\left(x - \frac{3}{2}\right)$

Or, (2y - 5) = - (2x - 3)

Or 2x + 2y – 8 = 0

Or, x + y – 4 = 0

$\therefore$ Length of perpendicular from (0, 0) on L_1 is

$\left|\frac{0+0-4}{\sqrt{2}}\right| = 2\sqrt{2}$

$\therefore$ Length of the chord $= 2\sqrt{(a)^2 - (2\sqrt{2})^2}$

$= 2\sqrt{16 - (2\sqrt{2})^2}$

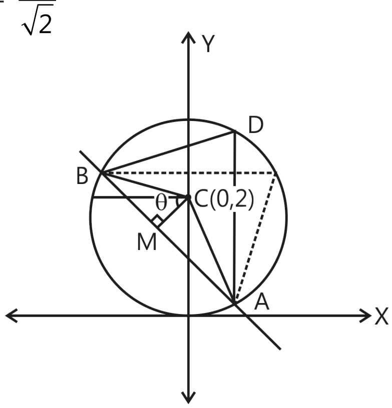
$= 4\sqrt{2}$

Sol 21: Equation of circle is $x^2 + y^2 - 4y = 0$

$\therefore$ Centre = (0, 2) & radius = 2

Perpendicular distance of center from the line x + y = 1 is

$\frac{0+2-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$



Let θ be the angle subtended at the circumference

$\therefore$ Angle subtended at circumference

$= \frac{1}{2}$ (Angle subtended at centre)

$\therefore \cos\theta = \frac{1}{2\sqrt{2}} \Rightarrow \theta = \cos^{-1}\frac{1}{2\sqrt{2}}$

Sol 22: Given $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$

$\Rightarrow (x^2 + y^2 - 2x - 8) - 2\lambda(y) = 0$ ....(i)

Let $S \equiv x^2 + y^2 - 2x - 8 = 0$ and ...(iii)

$L \equiv y = 0$ ...(iii)

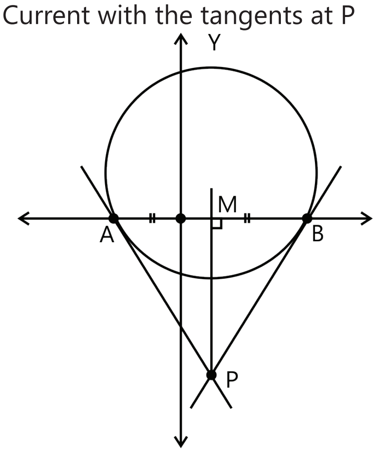
$\therefore$ The equation is represents a family of circles passing through the intersection of S = 0 & L = 0.

$\therefore$ On solving (ii) and (iii), we get

$x^2 - 2x - 8 = 0$

$\Rightarrow x = \frac{2 \pm \sqrt{4+32}}{2} = 4$ or -2

$\therefore$ The fixed point are A (4, 0) and B (-2, 0) from the diagram, the perpendicular bisector of AB is con. Current with the tangents at P



$\therefore M \equiv (1, 0)$

And Equation of line MP is x = 1 .....(iv)

$\therefore$ On solving (iv) with x + 2y + 5 = 0

We get 1 + 2y + 5 = 0

$\Rightarrow$ 2y + 6 = 0 $\Rightarrow y = \frac{-6}{2} = -3$

$\therefore P \equiv (1, -3)$

Centre of circle (i) is $C \equiv (1, \lambda)$

If P is the point of intersection of tangents then CB is perpendicular to BP

$\therefore \left(\frac{\lambda - 0}{1-4}\right) \times \left(\frac{0+3}{4-1}\right) = -1$

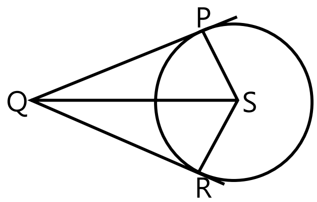
$\therefore \frac{\lambda}{-3} \times \frac{3}{3} = -1 \Rightarrow \lambda = 3$

$\therefore$ Equation of the required circle is

$x^2 + y^2 - 2x - 6y - 8 = 0$

Sol 23: Length of tangent = $\sqrt{S_{11}}$

$\therefore$ QP = $\sqrt{4^2 + 5^2 - 4^2 - 10 - 11}$ = 2



Area of PQRS = 2ΔPQS = 2 × $\frac{1}{2}$ × PS x QPPS

= Radius of circle = $\sqrt{2^2 + 1^2 + 11}$ = 4

$\therefore$ Area of PQRS = 4 × 2 = 8

Sol 24: The equation of any curve passing through the intersection of

$L_1 \equiv a_1x + b_1y + c_1 = 0$

$L_2 \equiv a_2x + b_2y + c_2 = 0$

$L_3 \equiv y = 0$ & $L_4 \equiv x = 0$ is $L_1 L_2 + \lambda L_3 L_4$

$\Rightarrow (a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + \lambda xy = 0$

where λ is a parameter

This curve represents a circle if coeff. of x^2 = coeff. of y^2

$\therefore a_1a_2 = b_1b_2$

Sol 25: Let any point on c_2 be (h, k)

Length of tangent from any point to circle

= $\sqrt{S_1}$

$\therefore l = \sqrt{h^2 + k^2 + 2gh + 2fk + c_1}$

Now since (h, k) satisfies circle 1

$\therefore h^2 + k^2 + 2gh + 2fk = -c$

$\therefore l = \sqrt{c_1 - c}$

Sol 26: The tangents to the these circle are equal in length

$\therefore$ The point is radical centre

The equation of radical axes are $S_1 - S_2 = 0$

$\therefore S_1 - S_2 = \left(\frac{3}{2} - 4\right)x - \frac{5}{2}y + \frac{7-9}{2} = 0$

$\Rightarrow 5x + 5y - 5 = 0 \Rightarrow x + y - 1 = 0$

and $S_1 - S_3 = 0 \Rightarrow -4x - y + 7 = 0$

$4x + y - 7 = 0$

$\therefore$ The radical centre is (2, –1)

Length of tangent = $\sqrt{S_1} = \sqrt{2^2 + 1^2 - 8 + 7}$ =2

Sol 27: Let (h, k) be the point on circle $x^2 + y^2 = a^2$

$\Rightarrow \therefore h^2 + k^2 = a^2$ ...(i)

Equation of chord of contact for $x^2 + y^2 = b^2$ is $hx + ky = b^2$ ...(ii)

As (ii) touches the circle $x^2 + y^2 = c^2$

$\therefore \left|\frac{-b^2}{\sqrt{h^2 + k^2}}\right| = c$

$\Rightarrow b^2 = ac$

$\therefore$ a, b and c are in G.P.

Sol 28: Let the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

The given circles are

$x^2 + y^2 + 3x - 5y + 6 = 0$ ... (ii)

and $x^2 + y^2 - 7x + \frac{29}{4} = 0$ ... (iii)

Now 1, 2 & 1, 3 are orthogonal

$\therefore 2g\frac{3}{2} + 2f\frac{-5}{2} = c + 6$

$3g - 5f = c + 6$ & $2g \times \frac{-7}{2} + 2f \times 0 = c + \frac{29}{4}$

$\Rightarrow -7g = c + \frac{29}{4}$

$\therefore 10g - 5f = \frac{-5}{4}$

$\therefore 8g - 4f = -1$

Equation of circle is

$x^2 + y^2 + 2gx + \frac{(8g+1)}{2}y + c = 0$

The centre lies on the line

$3x + 4y + 1 = 0$

$\Rightarrow 3(-g) - 4\frac{(8g+1)}{4} + 1 = 0$

$\Rightarrow -11g = 0$

$\Rightarrow g = 0,\ f = \dfrac{1}{4}$ and $c = -\dfrac{29}{4}$

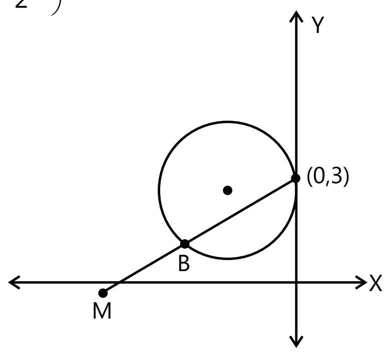
$\therefore$ Equation of circle is

$4x^2 + 4y^2 + 2y - 29 = 0$

Sol 29: Given equation of circle is $x^2 + 4x + (y-3)^2 = 0$

Let $M \equiv (h, k)$ $\quad\quad \therefore B \equiv \left(\dfrac{0+h}{2}, \dfrac{3+k}{2}\right)$

$\therefore B \equiv \left(\dfrac{h}{2}, \dfrac{3+k}{2}\right)$



As point B lies on the circle

$$\therefore \frac{h^2}{4} + 4 \times \frac{h}{2} + \left(\frac{3+k}{2} - 3\right)^2 = 0$$

$$\Rightarrow \frac{h^2}{4} + 2h + \frac{k^2}{4} + \frac{9}{4} - 2\frac{k}{2} \times \frac{3}{2} = 0$$

$$\Rightarrow h^2 + k^2 + 8h - 6k + 9 = 0$$

$\therefore$ The value of point B is

$x^2 + y^2 + 8x - 6y + 9 = 0$

Sol 30: Let (h, k) be middle points

Equation of chord through (h, k) is

$xh - (x + h) + yk = h^2 - 2h + k^2$ ... (i)

As the chord given by equation (i) passes through (0, 0)

$\therefore$ On substituting, x = 0 and y = 0, we get

$-h = h^2 - 2h + k^2$

$\therefore$ Locus of midpoint is $x^2 - x + y^2 = 0$

Sol 31: Given, OM = a and OP = b

From the diagram,

$\angle PRQ = 90°$

And PR = QR

$\angle QPR = \angle PQR = 45°$

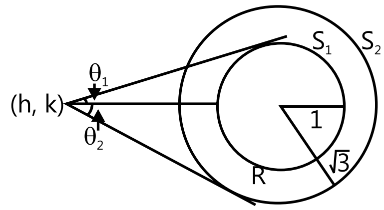
$\angle OPR = 90° - \angle QPR = 45°$

$\therefore$ In ΔOMP, $\sin 45° = \dfrac{OM}{OP}$

$\Rightarrow \dfrac{1}{\sqrt{2}} = \dfrac{a}{b}$

$\Rightarrow b = \sqrt{2}\,a$.

Sol 32: According to condition



$\theta_1 + \theta_2 = 90°$

$\therefore \tan\theta_1 \tan\theta_2 = 1$

$$\tan\theta_1 = \frac{r_1}{\text{Length of tangent}} = \frac{1}{\sqrt{h^2 + k^2 - 1}}$$

$$\tan\theta_2 = \frac{\sqrt{3}}{\sqrt{h^2 + k^2 - 3}}$$

According to condition –

$\therefore\ 3 = (h^2 + k^2 - 1)(h^2 + k^2 - 3)$

$3 = (h^2 + k^2)^2 - 4(h^2 + k^2) + 3$

$\therefore\ h^2 + k^2 = 0$

or $h^2 + k^2 = 4$

Now $h^2 + k^2 \neq 0$ as no tangent will be possible.

$\therefore$ The locus of point is a circle

Sol 33: Let the other end of diameter be (h, k)

$\therefore$ Equation of circle is

$(x - a)(x - h) + (y - b)(y - k) = 0$

$\therefore$ Center $\equiv \left(\dfrac{a+h}{2}, \dfrac{b+k}{2}\right)$

Since the circle touches the x-axis

$\therefore$ | y-coordinate| = radius

$$\Rightarrow \left|\frac{b+k}{2}\right| = \sqrt{\left(\frac{a+h}{2}\right)^2 + \left(\frac{b+k}{2}\right)^2 - (ah + bk)}$$

$$\therefore \left(\frac{a+h}{2}\right)^2 = (ah + bk)$$

$\therefore$ Locus of point is

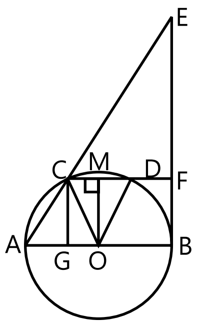
$x^2 + 2ax + a^2 = 4ax + 4by$

$(x - a)^2 = 4by$

Sol 34: Let G be perpendicular from C on AB

And M be midpoint of CD

Let radius = R



$\therefore MO^2 + MD^2 = OD^2$ (O is centre)

$MO^2 = R^2 - \frac{R^2}{4}$

$MO = \frac{\sqrt{3}R}{4}$

$\Rightarrow CG = \frac{\sqrt{3}R}{4}$

$AG = AO - GO = AO - CM = R - \frac{R}{2} = \frac{R}{2}$

$AC^2 = AG^2 + GC^2 = \frac{3R^2}{4} + \frac{R^2}{4}$

$\therefore AC = R$

$\therefore \frac{AE}{AC} = \frac{AB}{AG}$ (As Δ AEB ~ Δ ACG)

$\Rightarrow \frac{AE}{R} = \frac{AB}{\frac{R}{2}} \Rightarrow AE = 2AB$

Exercise 2

Single Correct Choice Type

Sol 1: (D) The centers are A = (2, 3) ; B = (–1, –2) ; C = (5, 8)

$\therefore$ Slope of AB = $\frac{3-(-2)}{2-(-1)} = \frac{5}{3}$

and slope AC = $\frac{8-3}{5-2} = \frac{5}{3}$

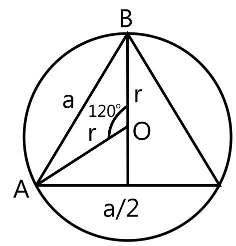
$\therefore$ The three points are collinear

Sol 2: (B) $S \equiv x^2 + y^2 + \lambda x + \frac{\lambda^2}{2} = 0$

Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{2}} = \sqrt{-\frac{\lambda^2}{4}}$

$\therefore$ Radius is not defined for any real value of λ

Sol 3: (A) For an equilateral triangle inscribed in circle of radius r, in Δ OAB using cosine rule, we get



$\cos 120° = \frac{r^2 + r^2 - a^2}{2r^2}$

$\Rightarrow -r^2 = 2r^2 - a^2$

$\Rightarrow a = \sqrt{3}\, r$

Area of equilateral triangle

$= \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4} \times (\sqrt{3}\, r)^2 = \frac{3\sqrt{3}}{4} r^2$

Radius of given circle = $\sqrt{g^2 + f^2 - c} = 1$

$\therefore A = \frac{3\sqrt{3}}{4}$

Sol 4: (B) Let the centre of circle be (– h, 0) where h > 0

Radius = 5

$\therefore$ Equation of circle is $(x + h)^2 + y^2 = 25$

It passes through the point (2, 3)

$\therefore (h + 2)^2 = (4)^2 \Rightarrow h = 2$ or $h = -6$

But $h > 0 \Rightarrow h = 2 \Rightarrow (2+h)^2 + 9 = 25 \Rightarrow h = 2$ or -6

$\therefore$ Equation of the circle is $x^2 + y^2 + 2x - 21 = 0$

$\therefore$ Intercept made on y-axis = $2\sqrt{f^2 - c} = 2\sqrt{21}$

Sol 5: (A) $S_1 : x^2 + y^2 = 1$

$S_2 : x^2 + y^2 - 2x - 6y = 6$

$S_3 : x^2 + y^2 - 4x - 12y = 9$

$r_1 = 1$; $r_2 = \sqrt{1^2 + 3^2 + 6} = 4$; $r_3 = \sqrt{2^2 + 6^2 + 9} = 7$

$\therefore r_1, r_2, r_3$ are in A.P.

Sol 6: (D) $S_1: x^2 + y^2 + 2\lambda x + 4 = 0$

$S_2: x^2 + y^2 - 4\lambda x + 8 = 0$

Since both represent real circles

$\therefore r_1 \geq 0$ & $r_2 \geq 0$

$\therefore \lambda^2 - 4 \geq 0 \therefore \lambda \leq -2$ or $\lambda \geq 2$ ... (i)

$\therefore 4\lambda^2 - 8 \geq 0 \therefore \lambda \leq -\sqrt{2}$ or $\lambda \geq \sqrt{2}$ ... (ii)

From 1, 2 λ $(-\infty, -2] \cup [2, \infty)$

All of these lie within the range

Sol 7: (D) $s = x^2 + y^2 + 16x - 24y + 183 = 0$

Centre $\equiv (-8, 12)$ Radius = 5

Let (x_1, y_1) be the image of (–8, 12) w.r.t. to the line

$4x + 7y + 13 = 0$

$$\therefore \frac{x_1 - (-8)}{4} = \frac{y_1 - 12}{7}$$

$$= \frac{-2\{4 \times (-8) + 7 \times 12 + 13\}}{4^2 + 7^2}$$

$$\frac{x_1 + 8}{4} = \frac{y_1 - 12}{7} = -2$$

$x_1 = -16, y_1 = -2$

Equation of required circle is

$(x + 16)^2 + (y + 2)^2 = 5^2$

$x^2 + y^2 + 32x + 4y + 235 = 0$

Sol 8: (B) Equation of circle is

$(x - 0)(x - a) + (y - 1)(y - b) = 0$ ...(i)

Let the circle given by eq. (i) cut the x-axis at (h, 0)

$h(h - a) + b = 0$

$h^2 - ah + b = 0$

The abscissa are roots of equation $x^2 - ax + b = 0$

Sol 9: (C) $x = 2y - 10$ & $x^2 + y^2 = 100$

$\Rightarrow 4y^2 - 40y + y^2 = 0$

$\Rightarrow 5y(y - 8)\ 0$

$\therefore y = 8$ (as point lies in 1st quadrant & $x = +6$)

The line perpendicular to $x - 2y + 10 = 0$ passing through (6, 8) is $(y - 8) = -2(x - 6)$

$2x + y = 20$

It cuts the y-axis at (0, 20)

Sol 10: (C) Let equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

Let $\left(x, \frac{1}{x}\right)$ be a point on the circle.

$\therefore x^4 + 2gx^3 + cx^2 + 2fx + 1 = 0$

$$\Rightarrow abcd = \frac{1}{1} = 1$$

Sol 11: (A) Circumradius $R = \frac{abc}{4\Delta}$, where Δ is the area of a triangle

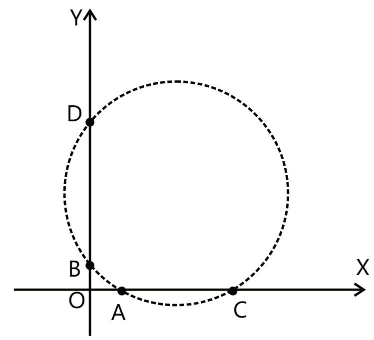
$$\Rightarrow R = \frac{12 \times 12 \times 6}{4 \times \left(\frac{1}{2} \times 6 \times \text{height}\right)}$$

$$\text{Height} = \sqrt{12^2 - 3^2} = 3\sqrt{15}$$

$$\therefore R = \frac{12 \times 6}{3\sqrt{15}} = \frac{8\sqrt{15}}{5}$$

Sol 12: (D) Given, ac = bd

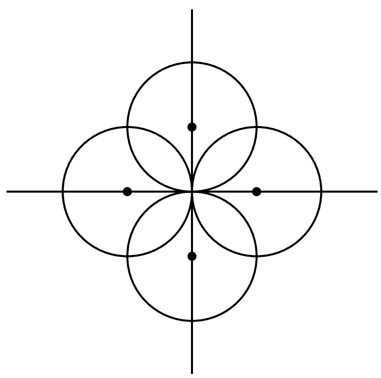
$\Rightarrow AO \times OC = OB \times OD$



This is true in case of circle and two secants

$\therefore$ A, B, C and D lie on a circle.

Sol 13: (C)



Since the centres lie on co-ordinate axes

The centre are (1, 0), (–1, 0), (0, 1) and (0, –1)

Consider two circles with centre (1, 0) & (0, 1)

Their point of intersection will lie on the line y = x

Putting y = x in $(x-1)^2 + y^2 = 1$

$\Rightarrow 2x^2 - 2x = 0$

$\Rightarrow$ x = 1 & y = 1 (ignoring x = y = 0)

(1, 1) is the point

By symmetry the other 3 points are

(1, –1) (–1, 1) (–1, –1).

It is a square of side 2 units

Area = 4 sq. units

Sol 14: (A) The y co-ordinate = 2, centre = (h, 2) & radius = 2

On using the condition of tangency on $y = \frac{x}{2}$,

we get $\frac{2\times 2 - h}{\sqrt{5}} = \pm 2$

$\Rightarrow h = 4 \pm 2\sqrt{5}$

But h > 0

x coordinate is $4 + 2\sqrt{5}$.

Sol 15: (B) Let the midpoint of chord be (h, k)

$\therefore$ Equation of chord is $T = S_1$

$\Rightarrow xh + 2(x + h) + yk - 3(y + k) + 9$

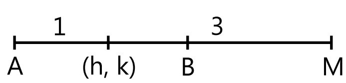
$= h^2 + 4h + k^2 - 6k + 9$

Since (0, 3) lies on this chord

$2h + 3k - 3(3+k) = h^2 + 4h + k^2 - 6k$

Locus of midpoint is

$h^2 + 2h + k^2 - 6k + 9 = 0$



$\therefore$ Let M be (x , y)

$$(h, k) = \left(\frac{3\times 0 + x}{4}, \frac{9+y}{4}\right)$$

Substituting in 1 we get locus of M.

$$\therefore \left(\frac{x}{4}\right)^2 + \frac{2(x)}{4} + \left(\frac{y+9}{4}\right)^2 - \frac{6\times(y+9)}{4} + 9 = 0$$

$\Rightarrow x^2 + y^2 + 8x - 6y + 81 - 216 + 144 = 0$

$\Rightarrow x^2 + 8x - (y-3)^2 = 0$

Alternate:

Since, $\frac{AM}{AB} = 2 \Rightarrow \frac{AB}{BM} = 1$

Let M be (h, k)

Then, $B \equiv \left(\frac{h}{2}, \frac{k+3}{2}\right)$

Which lies on Circle.

Substitute to get the required Locus.

Sol 16: (C) P = (0, 5)

$S_1 = x^2 + y^2 + 2x - 4 = 0$

$S_2 = x^2 + y^2 - y + 1 = 0$

$L_1 = \sqrt{25-4} = \sqrt{21}$

$L_2 = \sqrt{21}$

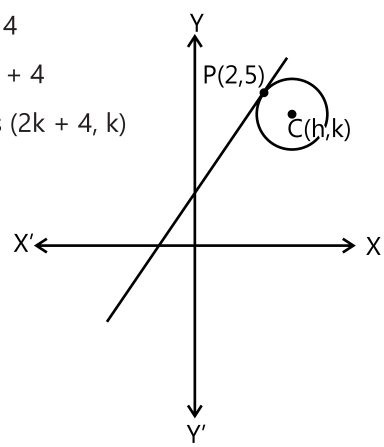
$\therefore L_1 = L_2$

Sol 17: (A) Let centre of circle be (h, k)

$\therefore h - 2k = 4$

$\Rightarrow h = 2k + 4$

$\therefore$ Centre is (2k + 4, k)



Now CP $\perp$ tangent

$$\therefore \frac{5-k}{2-2k-4} \times 2 = -1$$

$\therefore 5 - k = \frac{(2k+2)}{2}$

$\therefore 5 - k = k + 1$

$\therefore k = 2$

Center is (8, 2)

Radius = $\sqrt{(8-2)^2 + (2-5)^2} = 3\sqrt{5}$

Sol 18: (D) Let circle be $S \equiv x^2+y^2+2gx+2fy + c = 0$

$S_1 \equiv x^2 + y^2 = 1$

$S_2 \equiv x^2 + y^2 + 2x - 3 = 0$

$S_3 \equiv x^2 + y^2 + 2y - 3 = 0$

$\Rightarrow S - S_1 = 0$ is the equation of chord of contact & it passes through centre of S_1

$\Rightarrow 2gx + 2fy + c + 1 = 0$

Satisfying (0, 0) $\Rightarrow$ c = – 1,

Similarly $S - S_2 = 0$

$\Rightarrow (2g - 2)x + 2fy + 2 = 0$

Satisfying (-1, 0), we get $2 - 2g + 2 = 0$

$\Rightarrow g = 2$

Similarly, $S - S_3 = 0$

$\Rightarrow (2gx + (2f - 2)y + 2 = 0$

(Satisfying (0, –1), we get $\Rightarrow f = 2$

$\therefore$ Centre is (–2, –2)

Sol 19: (C) Let tangent from origin be y = mx

Using the condition of tangency, we get

$$\Rightarrow \frac{7m+1}{\sqrt{m^2+1}} = 5$$

$(7m + 1)^2 = 25(m^2 + 1)$

$\Rightarrow 24m^2 + 14m - 24 = 0$

$\Rightarrow 12m^2 + 7m - 12 = 0$

$\Rightarrow 12m^2 + 16m - 9m - 12 = 0$

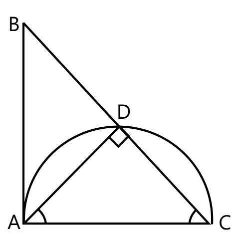
$(4m - 3)(3m + 4)$

$\therefore m = \frac{3}{4}$ and $m = -\frac{4}{3}$

The angle between tangents = $\frac{\pi}{2}$

Sol 20: (D) Since A, D, C lies on the circle with AC as the diameter

AD $\perp$ DC



$\therefore \Delta ADC \sim \Delta ABC$

$$\Rightarrow \frac{AC}{BC} = \frac{AD}{AB} \quad ...(i)$$

Also, $BC^2 = \sqrt{AB^2 + AC^2}$ ...(ii)

$$AC^2 = (AB^2 + AC^2)\frac{AD^2}{AB^2}$$

[From (i) and (ii)]

$$AC^2 = \frac{AB^2AD^2}{AB^2 - AD^2}$$

$$AC = \frac{AB.AD}{\sqrt{AB^2 - AD^2}}$$

Sol 21: (A) $x^3 + y^3 + 3xy - 1 = 0$

$\Rightarrow (x + y)^3 - 3xy(x + y) + 3xy - 1 = 0$

$\Rightarrow (x + y)^3 - 3xy(x + y - 1) - 1^3 = 0$

$\Rightarrow (x + y)^3 - 1^3 = 3xy(x + y - 1)$

$\Rightarrow (x + y - 1)\{(x + y)^2 + (x + y) + 1\} - 3xy(x + y - 1) = 0$

We get,

$\therefore (x + y - 1)\{(x + y)^2 + (x + y) + 1 - 3xy\} = 0$

$(x + y - 1)(x^2 + y^2 - xy + x + y + 1) = 0$

For the curve $x^2 + y^2 - xy + x + y + 1 = 0$

$ab - h^2 = 1 - \frac{1}{4} > 0$

and

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}$$

$$= 1 \times \left(1 - \frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{3}{4} - \frac{3}{4} = 0$$

$\therefore$ It is a point

Previous Years' Questions

Sol 1: (D) Equation of circle passing through a point (x_1, y_1) and touching the straight line L, is given by

$(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$

$\therefore$ Equation of circle passing through (0, 2) and touching x=0

$\Rightarrow (x - 0)^2 + (y - 2)^2 + \lambda x = 0$ .....(i)

Also, it passes through (-1, 0)

$\Rightarrow 1 + 4 - \lambda = 0 \;\; \lambda \Rightarrow 5$

$\therefore$ Eq. (i) becomes,

$x^2 + y^2 - 4y + 4 + 5x = 0$

$\Rightarrow x^2 + y^2 + 5x - 4y + 4 = 0,$

For x-intercept put y = 0

$\Rightarrow x^2 + 5x + 4 = 0$

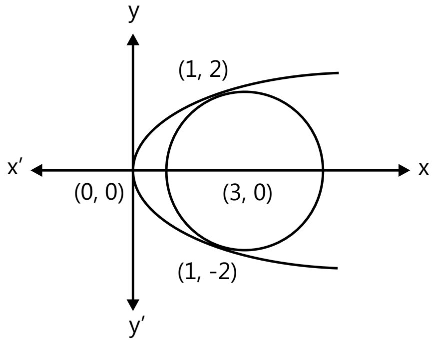
$(x+1)(x+4) = 0$

$\therefore\ x = -1, -4$

Sol 2: (B) For the point of intersection of the two given curves

$C_1 : y^2 = 4x$

and $C_2 : x^2 + y^2 - 6x + 1 = 0$



We have, $x^2 + 4x - 6x + 1 = 0$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

$\Rightarrow x = 1$ (equal real roots)

$\Rightarrow y = 2, -2$

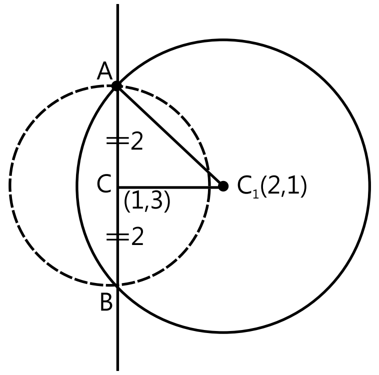
Thus, the given curves touch each other at exactly two point (1, 2) and (1, -2).

Sol 3: (C) Here radius of smaller circle, AC– $\sqrt{1^2 + 3^2 - 6} = 2$ Clearly, from the figure the radius of bigger circle

$r^2 = 2^2 + [(2-1)^2 + (1-3)^2]$

$r^2 = 9$

$\Rightarrow r = 3$

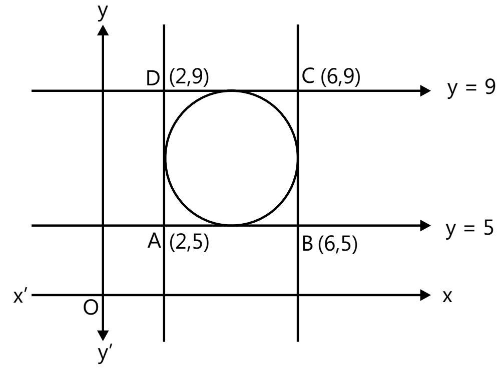


Sol 4: (A) Given, circle is inscribed in square formed by the lines

$x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$

$\Rightarrow x = 6$ and $x = 2, y = 5$ and $y = 9$

Which could be plotted as



Where ABCD clearly forms a square

$\therefore$ Centre of inscribed circle

= Point of intersection of diagonals

= Mid point of AC or BD

$= \left(\frac{2+6}{2}\right), \left(\frac{5+9}{2}\right) = (4, 7)$

$\Rightarrow$ Centre of inscribed circle is (4, 7)

Sol 5: (C) The line 5x-2y+6=0 meets

The y-axis at the point (0, 3) and therefore the tangent has to pass through the point (0, 3) and required length

$= \sqrt{x_1^2 + y_1^2 + 6x_1 + 6y_1 - 2}$

$= \sqrt{0^2 + 3^2 + 6(0) + 6(3) - 2} = \sqrt{25} = 5$

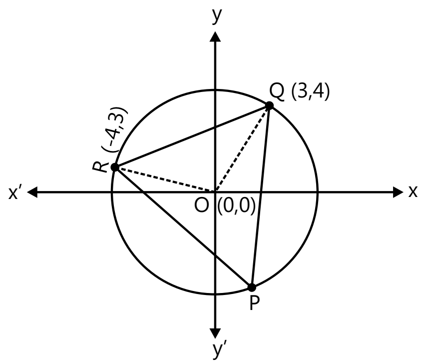
Sol 6: (A) Since, the given circles intersect orthogonally.

$2\left(g_1\ g_2 + f_1\ f_2\right) = G + C_2$

$\therefore \quad 2(-1)(0) + 2(-k)(-k) = 6 + k$

$\Rightarrow \quad 2k^2 - k - 6 = 0 \ \Rightarrow\ k = -\frac{3}{2}, 2$

Sol 7: (C) Let O is the point at centre and P is the point at circumference. Therefore, angle QOR is double the angle QPR. So it is sufficient to find the angle QOR.



Now, slope of OQ, m1 = 4/3, slope of OR, m_2 = −3/4

Here, $m_1 m_2 = -1$

Threfore, $\angle QOR = \pi/2$

Which implies that $\angle QPR = \pi/4$

Sol 8: (B) Given, $x^2 + y^2 = 4$

Centre $\equiv C_1 \equiv (0,0)$ and $R_1 = 2$

Again, $x^2 + y^2 - 6x - 8y - 24 = 0$, then $C_2 \equiv (3,4)$

and $R_2 = 7$ again, $C_1C_2 = 5 = R_2 - R_1$

Since, the given circles touch internally therefore, they can have just one common tangent at the point of contact.

Sol 9: Since, the tangents are perpendicular.

So, locus of perpendicular tangents to circle

$x^2 + y^2 = 169$ is a director circle having equation

$x^2 + y^2 = 338$

Sol 10: The equation of circle having tangent 2x+3y+1=0 at (1, -1)

$\Rightarrow (x-1)^2 + (y+1)^2 + \lambda(2x+3y+1) = 0$

$x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2) + (\lambda + 2) = 0$ ....(i)

Which is orthogonal to the circle having end point of diameter (0, -1) and (-2, 3)

$\Rightarrow x(x+2) + (y+1)(y-3) = 0$

or $x^2 + y^2 + 2x - 2y - 3 = 0$ ...(ii)

$\therefore \frac{2(2\lambda - 2)}{2} \cdot 1 + \frac{2(3\lambda + 2)}{2}(-1) = \lambda + 2 - 3$

$\Rightarrow 2\lambda - 2 - 3\lambda - 2 = \lambda - 1$

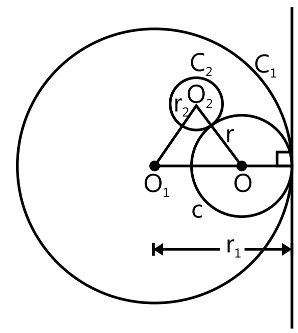
$\Rightarrow 2\lambda = -3 \Rightarrow \lambda = -3/2$

$\therefore$ From Eq. (i) equation of circle,

$2x^2 + 2y^2 - 10x - 5y + 1 = 0$

Sol 11: Let the given circles C_1 and C_2 have centres O_1 and O_2 and radii r_1 and r_2 respectively.

Let the variable circle C touching C_1 internally, C_2 externally have a radius r and centre at O



Now, $OO_2 = r + r_2$ and $OO_1 = r_1 - r$

$\Rightarrow OO_1 + OO_2 = r_1 + r_2$

Which is greater than

O_1O_2 as $O_1O_2 < r_1 + r_2$ ($\because C_2$ lies inside C_1)

$\Rightarrow$ Locus of O is an ellipse with foci O_1 and O_2

Alternate Solution

Let equations of C_1 be $x^2 + y^2 = r_1^2$ and of C_2 be $(x-a)^2 + (y-b)^2 = r_2^2$

Let centre C be (h, k) and radius r, then by the given condition

$\sqrt{(h-a)^2 + (k-b)^2} = r + r_2$ and $\sqrt{h^2 + k^2} = r_1 - r$

$\Rightarrow \sqrt{(h-a)^2 + (k-b)^2} + \sqrt{h^2 + k^2} = r_1 + r_2$

Required locus is

$\sqrt{(x-a)^2 + (y-b)^2} + \sqrt{x^2 + y^2} = r_1 + r_2$

Which represents an ellipse whose foci are at (a, b) and (0, 0).

Sol 12: Equation of any tangent to circle $x^2 + y^2 = r^2$ is

$x\cos\theta + y\sin\theta = r$ ....(i)

Suppose Eq. (i) is tangent to $4x^2 + 25y^2 = 100$

Or $\frac{x^2}{25} + \frac{y^2}{4} = 1$ at (x_1, y_1)

Then, Eq. (i) and $\frac{xx_1}{25} + \frac{yy_1}{4} = 1$ are identical

$\therefore \frac{x_1/25}{\cos\theta} = \frac{\frac{y_1}{4}}{\sin\theta} = \frac{1}{r}$

$\Rightarrow x_1 = \frac{25\cos\theta}{r}, y_1 = \frac{4\sin\theta}{r}$

The line (i) meet the coordinates axes in $A(r\sec\theta,0)$ and $\beta(0,r\operatorname{cosec}\theta)$. Let (h, k) be mid point of AB.

Then, $h=\frac{r\sec\theta}{2}$ and $k=\frac{r\operatorname{cosec}\theta}{2}$

Therefore, $2h=\frac{r}{\cos\theta}$ and $2k=\frac{r}{\sin\theta}$

$\therefore \quad x_1=\frac{25}{2h}$ and $y_1=\frac{4}{2k}$

As (x_1,y_1) lies on the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$

We get $\frac{1}{25}\left(\frac{625}{4h^2}\right)+\frac{1}{4}\left(\frac{4}{k^2}\right)=1$

$\Rightarrow \frac{25}{4h^2}+\frac{1}{k^2}=1$

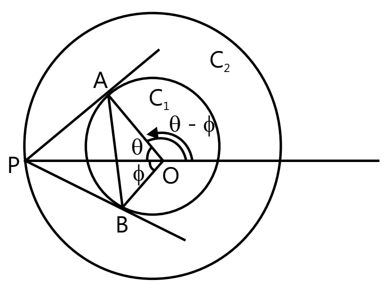
or $\quad 25k^2+4h^2=4h^2k^2$

Therefore, required locus is $4x^2+25y^2=4x^2y^2$

Sol 13: Let the coordinate of point P be $(2r\cos\theta,\ 2r\sin\theta)$

We have, OA = r, OP = 2r

Since, ΔOAP is a right angled triangle.



$\cos\phi=1/2$

$\Rightarrow \quad \phi=\pi/3$

$\therefore$ Coordinates of A and B are

$\{r\cos(\theta-\pi/3),r\sin(\theta-\pi/3)\}$ and

$\left\{r\cos\left(\theta+\frac{\pi}{3}\right)\right\}, r\sin\left(\theta+\frac{\pi}{3}\right)$

If p,q is the centroid of ΔPAB, then

$p=\frac{1}{3}[r\cos(\theta-\pi/3)+r\cos(\theta+\pi/3)+2r\cos\theta]$

$=\frac{1}{3}[r\{\cos(\theta-\pi/3)+\cos(\theta+\pi/3)\}+2r\cos\theta]$

$=\frac{1}{3}\left[r\left(2\cos\frac{\theta-\frac{\pi}{3}+\theta+\frac{\pi}{3}}{2}\cdot\cos\frac{\theta-\frac{\pi}{3}-\theta-\frac{\pi}{3}}{2}\right)+2r\cos\theta\right]$

$=\frac{1}{3}\left[r\left(2\sin\frac{\theta-\frac{\pi}{3}+\theta+\frac{\pi}{3}}{2}\cdot\cos\frac{\theta-\frac{\pi}{3}-\theta-\frac{\pi}{3}}{2}\right)+2r\sin\theta\right]$

$=\frac{1}{3}[r\{2\cos\theta\cos\pi/3\}+2r\cos\theta]$

$=\frac{1}{3}[r\cdot\cos\theta+2r\cos\theta]=r\cos\theta$

and $q=\frac{1}{3}[r\sin\left(\theta-\frac{\pi}{3}\right)+r\sin\left(\theta+\frac{\pi}{3}\right)+2r\sin\theta]$

$=\frac{1}{3}[r\{\sin\left(\theta-\frac{\pi}{3}\right)+\sin\left(\theta+\frac{\pi}{3}\right)\}+2r\sin\theta]$

$=\frac{1}{3}[r(2\sin\theta\cos\pi/3)+2r\sin\theta]$

$=\frac{1}{3}[r(\sin\theta)+2r\sin\theta]$

$=r\sin\theta$

Now, $(p,q)=(r\cos\theta,r\sin\theta)$ lies on x^2+y^2 $=r^2$, which is C_1

Sol 14: (A) Eq. of circle touching $x-a\times y$ at $(1,0)$ u

$(x-1)^2+(y-k)^2=k^2$

Circle passes through $(2,3)$, then

$(x-1)^2+(3-k)^2=k^2$

$1+9-6k+k^2=k^2$

$\Rightarrow 6k=10$

$\Rightarrow 2k=\frac{10}{3}$

Sol 15: (B) The eq. of circle touching the

$a-a\times\hat{u}$ at $(3,0)$ is

$(1-3)^2+(-2,-k)^2=k^2$

$\Rightarrow 4+4+4k+k^2=k^2$

$\Rightarrow 4k=-8$

$\Rightarrow k=-2$

Circle: $(x-3)^2(-2-k)^2=k^2$

Point $(5,-2)$

$(5,-3)^2+(-2+2)^2=2+2=4$

Only $(5,-2)$ lies on circle.

Sol 16: (D) $\frac{x^2}{10}+\frac{y^2}{9}=1$

to is $=\left(\pm\sqrt{7},0\right)$

Circle having cente as $\left(0,3\right)$

$x^2+\left(y-3\right)^2=\gamma^2$ passes through focus , then

$\left(\pm\sqrt{7}\right)^2+\left(0-3\right)^2=\gamma^2$

$7+9=\gamma^2$

$\Rightarrow \gamma^2=16$

$\Rightarrow x^2+\left(y-3\right)^2=16$

$\Rightarrow x^2+y^2-6y-7=0$

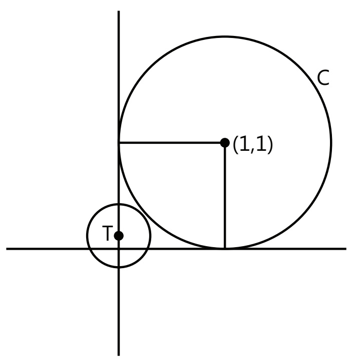
Sol 17: (D) If circles C and T touch each other externally then

$1+y=\sqrt{\left(1-0\right)^2+\left(1-y\right)^2}$

$\Rightarrow \left(1+y\right)^2=1+\left(1-y^2\right)$

$\Rightarrow 1^2+y^2+2y=1+1+y^2-2y$

$\Rightarrow y=\frac{1}{4}$



Sol 18: (B) $\left(x-2\right)^2+\left(y-3\right)^2=25$

$\left(x+3\right)^2+\left(y+9\right)^2=64$

$C_1C_2=\sqrt{\left(2+3\right)^2+\left(3+9\right)^2}$

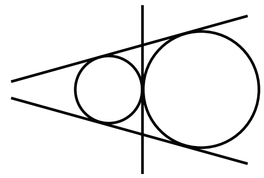
$=\sqrt{25+144}$

$=\sqrt{169}=13$

$r_1+r_2=3+5=13$

$\Rightarrow r_1+r_2=c_1\ c_2$

Circles touch each other externally therefrom, three tangents are possible



Sol 19: (C) $x^2+y^2-8x-8y-4=0$

$\left(x-4\right)^2+\left(y-4\right)^2=36$

Circles touch each other exotically

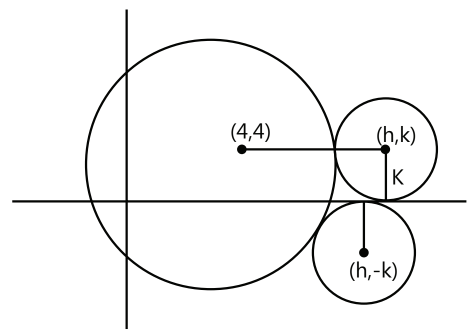
$k+6=\sqrt{\left(n-4\right)^2+\left(k-4\right)^2}$

$k^2+36+12k=h^2+16-3h+k^2-3k+16$

$\Rightarrow h^2-3h-9=20k$

$\Rightarrow x^2-3x-20y-4=0$

If $y<0$



$\left(-k+6\right)^2=\left(n-4\right)^2+\left(k-4\right)^2$

$\Rightarrow h^2-8h+4k-4=0$

$\Rightarrow x^2-8x+4y-4=0$

Locus is Parabola

JEE Advanced/Boards

Exercise 1

Sol 1: The equation of line through origin is y = mx

Let point on circle be (h_1, mh_1) and (h_2, mh_2)

S = x^2 + y^2 – 8x – 6y + 24 = 0

O = origin

(i) The equation of chord of S whose mid-point is (h, k) is

hx+ ky – 4(x + h) – 3(y + k) + 24

= h^2 + k^2 – 8h – 6k + 24

Since it passes through origin

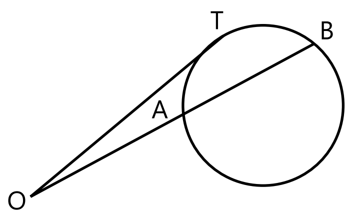
$\therefore -4h - 3k = h^2 + k^2 - 8h - 6k$

$\therefore$ Locus of point is

$x^2 + y^2 - 4x - 3y = 0$

(ii) OP = $\sqrt{OA \times OB}$

It is a known property that



OA × OB = $OT^2 = OP^2$

$\therefore$ OP = OT = constant k

OT = $\sqrt{S_{(0,0)}} = \sqrt{24}$

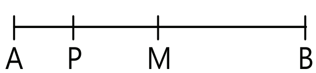
$\therefore$ The locus of P is the circle of radius $\sqrt{24}$ and centre = origin

$\Rightarrow x^2 + y^2 = 24$ is the locus of P

(iii) OP = $\dfrac{2OA.OB}{OA+OB} = \dfrac{OA.\ OB}{OM}$

$\therefore$ OP × OM = OA × OB

$\therefore$ A and M are harmonic conjugates of P & B



$\therefore \dfrac{AM}{PM} = \dfrac{AB}{MB} \Rightarrow \dfrac{AM}{PM} = 2$

$\therefore$ P is mid-point of A & M

$\therefore$ Locus of P:

$x^2 + y^2 - 8x - 6y + 24 - (x^2 + y^2 - 4x - 3y) = 0$

$\therefore 4x + 3y = 24$ is locus of P

Sol 2: Radius of given circle= $\sqrt{4+2-C} = \sqrt{6-C}$

$r = \sqrt{2}\, r_1$ and $r_1 = \sqrt{2}\, r_2$, $r_2 = \sqrt{2}\, r_3$

Sum of radii of all circles

$= r + \dfrac{r}{\sqrt{2}} + \dfrac{r}{2} + \ldots\ldots\ldots\ldots = \dfrac{r}{1-\dfrac{1}{\sqrt{2}}} \Rightarrow \dfrac{r}{1-\dfrac{1}{\sqrt{2}}} = 2$

$\therefore r = 2 - \sqrt{2}$

$\Rightarrow \sqrt{6-C} = 2 - \sqrt{2} \Rightarrow 6 - C = 4 + 2 - 4\sqrt{2}$

$\therefore C = 4\sqrt{2} = \sqrt{32} \Rightarrow n = 32$

Sol 3: Equation of common chord

$= x^2 + y^2 + 4x + 22y + a-(x^2 + y^2 - 2x + 8y - b)$

$= 6x + 14y + (a + b) = 0$

Now centre of second circle lies on this

$\therefore 6 \times 1 + 14 \times (-4) + (a + b) = 0$

$\therefore (a + b) = 50$

Now a , b > 0

$\therefore$ AM > GM

$\Rightarrow \dfrac{a+b}{2} > \sqrt{ab}$

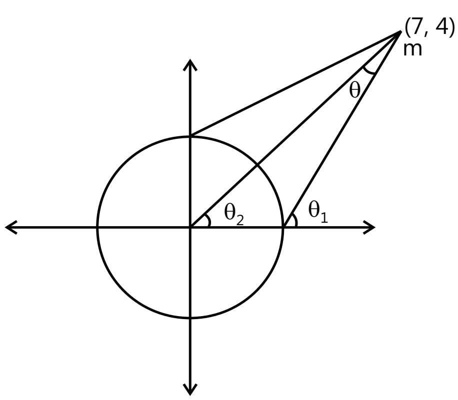
$\therefore 25 > \sqrt{ab}$

$ab < 625$

Sol 4: $x^2 + y^2 = 1$

$Z = \dfrac{y-4}{x-7}$

In this the slope from the point (7, 4); $\tan\theta_2 = \dfrac{4}{7}$



$\tan\theta = \tan(\theta_1 - \theta_2) = \dfrac{m - \tan\theta_2}{1 + m\tan\theta_2}$

$\left|\dfrac{r}{\ell_{\text{tangent}}}\right| = \dfrac{m - 4/7}{1 + m \times \dfrac{4}{7}}$

$\Rightarrow \pm\dfrac{1}{8} = \dfrac{m - 4/7}{1 + \dfrac{4m}{7}} = \dfrac{7m-4}{7+4m}$

$\therefore M = \dfrac{3}{4}$ and $m = \dfrac{5}{12}$

$\therefore 2M + 6m = \dfrac{2 \times 3}{4} + \dfrac{6 \times 5}{12} = 4$

Sol 5: The radical axis of 2 circles is

$\left(2g-\frac{3}{2}\right)x + (2f-4)y = 0$

Centre of the given circle = (–1, 1)

and radius = 1

Since it is a tangent to the circle

$\Rightarrow \dfrac{(2f-4)-\left(2g-\frac{3}{2}\right)}{\sqrt{\left(2g-\frac{3}{2}\right)^2+(2f-4)^2}} = 1$

$\Rightarrow (2f-4)^2 + \left(2g-\frac{3}{2}\right)^2 + 2(2f-4)\left(2g-\frac{3}{2}\right)$

$= \left(2g-\frac{3}{2}\right)^2 + (2t-4)^2$

$\therefore (2f-4)\left(2g-\frac{3}{2}\right) = 0$

$\therefore$ Either f = 2 or $g = \frac{3}{4}$

Sol 6: The line passing through points A(3, 7) and B(6, 5) is 2x + 3y – 27 = 0

The family of circles passing through these points is

$(x-3)(x-6) + (y-7)(y-5) + \lambda(2x+3y-27) = 0$

$\Rightarrow x^2 - 9x + 18 + y^2 - 12y + 35 + \lambda(2x+3y-27) = 0$

$\therefore$Chord of contact = $s_1 - s_2$

$\Rightarrow -5x - 6y + 50 + \lambda(2x+3y-27) = 0$

$\Rightarrow L_1 + \lambda L_2 = 0$

The point which passes through intersection of L_1 and L_2 is the point of intersection of all λ

5x + 6y = 50

2x + 3y = 27

$\therefore x = 2$ & $y = \frac{23}{3} \therefore P = \left(2,\frac{23}{3}\right)$

Sol 7: The locus of point of intersection of mutually perpendicular tangent is the director circle

$\therefore$Locus of point = $x^2 + y^2 = 8$

The equation of family of circle touch a given circles & at (x_1, y_1) is S + λ(L) where L = tangent

$x^2 + y^2 - 8 + \lambda(x \times 2 + y \times 2 - 8) = 0$

Now this passes through (1, 1)

$-6 + \lambda(-4) = 0$

$\lambda = \frac{-3}{2}$

Equation of circle is

$x^2 + y^2 - 8 - 3x - 3y + 12 = 0$

$\Rightarrow x^2 + y^2 - 3x - 3y + 4 = 0$

Sol 8: C: $x^2+y^2+y-1+k(x+y-1)=0$

It is the family of circle passing through points of intersection of a circle & L.

Putting x = 1 – y in C_1

We get $y^2 - 2y + 1 + fy^2 + y - 1 = 0$

$\Rightarrow 2y^2 - y = 0 \Rightarrow y = 0, \frac{1}{2}$ & $x = 1$ or $\frac{1}{2}$

$\therefore$ The point of intersection are A(1, 0) and $B\left(\frac{1}{2},\frac{1}{2}\right)$

The minimum value of radius is when point act as diameter

$\therefore r_{min} = \frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}\times\frac{1}{2}\sqrt{2} = \frac{1}{2\sqrt{2}}$

Sol 9: The equation of circle co-axial with 2 circle is $S_1 + \lambda S_2 = 0$

$2x^2 + 2y^2 - 2x + 6y - 3 +$

$\lambda(x^2 + y^2 + 4x + 2y + 1) = 0$

$= (2+\lambda)x^2 + (\lambda+2)y^2 + (4\lambda-2)x$

$+ (2\lambda+6)y + \lambda - 3 = 0$

$x_{centre} = \frac{2-4\lambda}{2(2+\lambda)} = \frac{1-2\lambda}{\lambda+2}$

$y_{centre} = \frac{-(\lambda+3)}{\lambda+2}$

Radical axis of the two circle is $s_1 - s_2 \equiv 5x - y + \frac{5}{2} = 0$

Centre lies on radical axis

$\therefore 5 \times \frac{(1-2\lambda)}{\lambda+2} + \frac{\lambda+3}{\lambda+2} + \frac{5}{2} = 0$

$\Rightarrow 10 - 20\lambda + 2\lambda + 6 + 5\lambda + 10 = 0$

$\Rightarrow 13\lambda = 26 \therefore \lambda = 2$

$\therefore$Equation of circle is $4x^2 + 4y^2 + 6x + 10y - 1 = 0$

Sol 10: $s_1 \equiv x^2 + y^2 - 4x - 6y - 12 = 0$

$s_2 \equiv x^2 + y^2 + 6x + 4y - 12 = 0$

$s_3 \equiv x^2 + y^2 - 2x - 4 = 0$

The circle passing through point of intersection of s_1 and s_2 is $s \equiv s_1 + \lambda s_2 = 0$

$\Rightarrow x^2 + y^2 - 4x - 6y - 12 + \lambda$

$(x^2 + y^2 + 6x + 4y - 12) = 0$

$\Rightarrow (\lambda + 1)x^2 + (\lambda + 1)y^2 + (6\lambda - 4)x$

$+ (4\lambda - 6)y - 12(\lambda + 1) = 0$

Since it is orthogonal to s_3

$\therefore 2gg_1 + 2ff_1 = c + c_1$

$\Rightarrow \dfrac{(6\lambda - 4)}{\lambda + 1} x - 1 + 0 = \dfrac{-12(\lambda + 1)}{(\lambda + 1)} - 4$

$\therefore 4 - 6\lambda = -16(\lambda + 1)$

$10\lambda = -20$

$\lambda = -2$

$s \equiv -x^2 - y^2 - 16x - 4y + 12 = 0$

$\therefore x^2 + y^2 + 16x + 14y - 12 = 0$

Sol 11: Let $s = x^2 + y^2 + 2gx + 2fy + c = 0$

Now $(-g, -f)$ lies on $2x - 2y + 9 = 0$

$\Rightarrow -2g + 2f + 9 = 0$

and it is orthogonal to $x^2 + y^2 - 4 = 0$

$2g \times 0 + 2f \times 0 = c - 4$

$C = 4$

and $f = g - \dfrac{9}{2}$

$s = x^2 + y^2 + 2gx + 2\left(g - \dfrac{9}{2}\right)y + 4 = 0$

$s \equiv x^2 + y^2 - 9y + 4 + 2g(x + y)$

$\therefore$ It passes through point of intersection of S and L

Putting $x = -y$ in s

$2y^2 - 9y + 4 = 0 \Rightarrow 2y^2 - 8y - y + 4 = 0$

$\therefore y = \dfrac{1}{2}$ or $y = 4$ & $x = -\dfrac{1}{2}, -4$

$\therefore$ The points are $\left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$ & $(-4, 4)$

Sol 12: Let the equation of circle be

$x^2 + y^2 + 2gx + 2fy = 0$

(it passes through origin)

The line pair is

$xy - 3x + 2y - 6 = 0$

$x(y - 3) + 2(y - 3) = 0$

$(x + 2)(y - 3) = 0$

The centre is point of intersection of these two lines

$c \equiv (-2, 3)$

$g = 2$ and $f = -3$

$s \equiv x^2 + y^2 + 4x - 6y = 0$

$s_1 = x^2 + y^2 - kx + 2ky - 8 = 0$

Since s & s_1 are orthogonal

$\therefore 2gg_1 + 2ff_1 = 0 - 8$

$\Rightarrow 2(-k) + (-3) \times 2k = 0 - 8$

$\therefore k = 1$

Sol 13: Since the circle cuts co-ordinate axis orthogonally

$\therefore C \equiv (0, 0)$

$\therefore s \equiv x^2 + y^2 - a^2 = 0$

$s \equiv x^2 + y^2 - 14x - 8y + 64 = 0$

Since s & s_1 are orthogonal

$\therefore 2 \times 0 \times -7 + 2 \times 0 \times -4 = -a^2 + 64$

$\therefore a^2 = 64$

$\therefore s \equiv x^2 + y^2 - 64 = 0$

Sol 14: Let the given circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Let the circle orthogonal to the two circles be

$x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore 2gg_1 + 2ff_1 = c_1 + c$

and $2gg_2 + 2ff_2 = c + c_2$

$\Rightarrow 2g(g_1 - g_2) + 2f(f_1 - f_2) = c_1 - c_2$

Now the centre is $(-g, -f)$

$\therefore x = -g$ & $y = -f$ substituting instead of g & f

We get $2x(g_1 - g_2) + 2y(f_1 + f_2) = (c_1 - c_2)$

Which is the radical axis & (straight line)

The locus of centres of given s_1, s_2 is $s_1 - s_2 = 0$

$4x + 5x - 6y - 4y + 7 = 0$

$9x - 10y + 7 = 0$

Sol 15: Consider a point circle at $(-2, 7)$

$(x + 2)^2 + (y - 7)^2 = 0$

Now the equation a circle touching a circle at point is $s + \lambda L$

Where L is tangent to L

$\therefore s \equiv (x + 2)^2 + (y - 7)^2 + \lambda(x + y - 5) = 0$

$\Rightarrow x^2 + y^2 + (\lambda + 4)x + (\lambda - 14)y + (53 - 5\lambda) = 0$

$\therefore s_2 \equiv x^2 + y^2 + 4x - 6y + 9 = 0$

Since s & s_2 are orthogonal

$\therefore (\lambda + 4) + (\lambda - 14)\,x - 3 = 53 - 5\lambda + 9$

$\Rightarrow 4\lambda = 12 \Rightarrow \lambda = 3$

$\therefore$ Equation of circle

$s \equiv x^2 + y^2 + 7x - 11y + 38 = 0$

Sol 16: Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

(–6, 0) lies on the circle

$\therefore 36 - 12g + c = 0$ ... (i)

The power of (i, i) is 5

$\Rightarrow 1 + 1 + 2g + 2f + c = 5$

$\Rightarrow 2g + 2f + c = 3$ ... (ii)

S is orthogonal to

$x^2 + y^2 - 4x - 6y - 3 = 0$

$\Rightarrow 2g(-2) + 2f(-3) = c - 3$

$\therefore 4g + 6f + c - 3 = 0$ ... (iii)

From (ii) and (iii)

$2g + 2c = 6 \Rightarrow g + c = 3$ ... (iv)

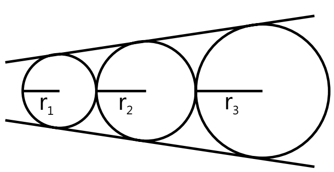
From i and iv g = 3

$\therefore c = 0 \quad \& \; f = \dfrac{-3}{2}$

$s \equiv x^2 + y^2 + 6x - 3y = 0$

Sol 17: Radius of largest circle = 18

Radius of smallest circle = 8



When 3 circle touching each other have direct common tangent

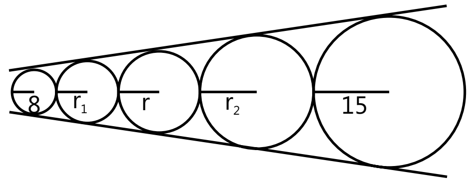
The radius of the middle circle is GM of radius of other 2 circles

$\therefore r_2^2 = r_1 r_3$

In the given problem

Let radius of middle circle be r,

2nd smallest circle be r_1 & 2nd largest circle be r_2



$\therefore r_1^2 = 8r$ & $r_2^2 = 18r$

$r^2 = r_1 r_2$

$\therefore r^4 = 8r \times 18r$

$r = \sqrt{8 \times 18} = 12$

Sol 18: The pair of lines is

$7x^2 - 18xy + 7y^2 = 0$

Since co-eff of x = coeff of y.

angle bisectors are

$(x - y) = 0$ & $x + y = 0$

Since the given circle lies in the 1st quadrant

$\therefore$ Our circle should also lie in the 1st quadrant

$\therefore$ Its centre should lie on y = x

Centre $\equiv$ (h , h)

Now $(x - h)^2 + (y - h)^2 = k^2$

Let y = mx be equation of tangent

$\dfrac{h - mh}{\sqrt{1 + m^2}} = R$

$\therefore R^2(m^2 + 1) = h^2(m^2 - 2m + 1)$

$\therefore (R^2 - h^2)m^2 + 2h^2 m + R^2 - h^2 = 0$

Comparing to pair of lines

We get $\dfrac{2h^2}{R^2 - h^2} = \dfrac{-18}{7}$

$14h^2 = -18R^2 + 18h^2$

$\therefore 4h^2 = 18R^2$

$\therefore h = \dfrac{3}{\sqrt{2}} R$ h is in 1st quadrant

Since C touches C_1

$= (R_1 - R)$ = distance between centres

$\therefore (4\sqrt{2} - R)^2 = \left(4 - \dfrac{3R}{\sqrt{2}}\right)^2 + \left(4 - \dfrac{3R}{\sqrt{2}}\right)^2$

$\Rightarrow 4\sqrt{2} - R = \pm \sqrt{2}\left(4 - \dfrac{3R}{\sqrt{2}}\right)$

$\therefore 4\sqrt{2} - R = -\sqrt{2}\left(4 - \frac{3R}{\sqrt{2}}\right)$

$8\sqrt{2} = \frac{3\sqrt{2}R}{\sqrt{2}} + R$

$8\sqrt{2} = 4R$

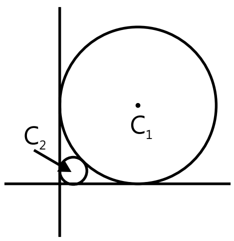
$R = 2\sqrt{2}$

$h = \frac{3 \times 2\sqrt{2}}{\sqrt{2}} = 6$

Equation is $(x-6)^2+(y-6)^2 = (2\sqrt{2})^2$

Sol 19: (r, p, q) (A) Centre of $C_1 \equiv (a, a)$ & radius = a for C_2 centre $\equiv (b, b)$ & radius = b

C_1 & C_2 cannot touch other internally



$\sqrt{(b-a)^2 + (b-a)^2} = (b + a)$

$\therefore \sqrt{2}(b - a) = (b + a)$

$\therefore (\sqrt{2} - 1)b = (\sqrt{2} + 1)a$

$\therefore \frac{b}{a} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = 3 + 2\sqrt{2}$

(B) Equation of

$C_1 \equiv x^2 + y^2 - 2ax - 2ay + a^2 = 0$

$C_2 \equiv x^2 + y^2 - 2bx - 2by + b^2 = 0$

C_1 & C_2 are orthogonal

$2(-a_x - b) + 2(-a_x - b) = a^2 + b^2$

$4ab = a^2 + b^2$

$a^2 - 4ab + b^2=0$

$\frac{b}{a} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2}$

$= 2 \pm \sqrt{3}$

But $b > a$ $\therefore \frac{b}{a} = 2 + \sqrt{3}$

(C) C_1 and C_2 intersect such that common chord is longest

$\therefore C_2$ bisects C_1

Equation of common chord is

$2(b - a)x + 2(b - a)y = b^2 - a^2$

$\therefore 2x + 2y = a + b$

It passes through (a, a)

$4a = a + b$

$b = 3a$

$\Rightarrow \frac{b}{a} = 3$

(D) C_2 passes through centre of C_1

$\therefore a^2 + a^2 - 2ab - 2ab + b^2 = 0$

$\Rightarrow b^2 - 4ab + 2a^2 = 0$

$\Rightarrow \left(\frac{b}{a}\right)^2 - 4\left(\frac{b}{a}\right) + 2 = 0$

$\therefore \frac{b}{a} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$

But $b > a$

$\therefore \frac{b}{a} = 2 + \sqrt{2}$

Sol 20: $y = x + 10$ & $y = x - 6$ are tangents

The centre of circle passes through

$y = x + \frac{(10-6)}{2} = y = x + 2$

Also radius, $= \frac{1}{2} \perp$ distance between lines

$= \frac{1}{2} \times \frac{c_1 - c_2}{\sqrt{a^2+b^2}} = \frac{1}{2} \times \frac{16}{\sqrt{2}} = 4\sqrt{2}$

$\therefore$ Circle is $(x - h)^2 + (y - (h + 2))^2 = (4\sqrt{2})^2$

$h + k = 2h + 2 = a + b\sqrt{a}$

Since y – axis is tangent

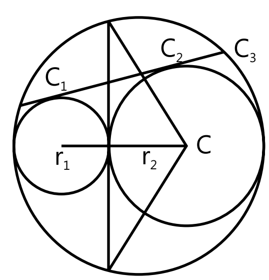
$\therefore h = \text{Radius}$

$\therefore h = 4\sqrt{2}$

and $h + k = 2h + 2 = 8\sqrt{2} + 2$

$\therefore a + b = 10$

Sol 21:



Since, centres of the Circle are collinear.

$\therefore$ Radius of bigger circle = $\frac{2r_1 + 2r_2}{2}$ = 14

Now distance of point of intersection from centre = R – $(2r_1)$ = 14 – 2 × 4 = 6 = d

Length of chord

= $2\sqrt{R^2 - d^2}$ = $2(14)^2 - (6)^2$= 4 $\sqrt{40}$ = 8 $\sqrt{10}$

m + n + p = 1 + 8 + 10 = 19

Sol 22: Equation of a circle passing through two points

$(x - 1)(x - 4) + (y - 7)(y - 8) + \lambda(L) = 0$

L passing through (4, 7) & (1, 8)

is $y - 8 = \frac{-1}{3}(x - 1)$

$3y + x - 25 = 0$

$\therefore$ $(x - 1)(x - 4) + (y - 7)(y - 8) + \lambda(3y + x - 25) = 0$

(5, 6) satisfies this equation

$\lambda = + \frac{(4+2)}{+2} = 3$

Equation of circle is $x^2 + y^2 - 2x - 6y - 15 = 0$

Let the points of intersection of tangent be (h , k)

chord of contact is

$hx + ky - (x + h) - 3(y + k) - 15 = h^2 + k^2 - 2h - 6h - 15$

$(h - 1)x + (k - 3)y + h + 3k - h^2 - k^2 = 0$

Now, $\frac{(h-1)}{h+3k-h^2-k^2} = \frac{5}{17}$ ... (i)

and $\frac{k-3}{h+3k-h^2-k^2} = \frac{1}{17}$

$\frac{h-1}{k-3} = 5 \Rightarrow h - 1 = 5(k - 3)$

$h = 5(K - 3) + 1$

Substituting in 1 we get k = 2

$\therefore$ h = – 4

$\therefore$ Point is (–4, 2)

Sol 23: The equation of circle which touches a given line at a point is

$(x-1)^2 + (y - 1)^2 + \lambda(2x - 3y + 1) = 0$

$\therefore x^2 - y^2 + 2(\lambda - 1)x - (2 + 3\lambda)y + \lambda + 2 = 0$

$R = \sqrt{13}$

$\therefore (\lambda - 1)^2 + \left(\frac{(2+3\lambda)}{2}\right)^2 - \lambda - 2 = 13$

$\therefore \lambda^2 = 4 \therefore \lambda = \pm 2$

$\therefore$ Equation of circles are

$x^2 + y^2 + 2x - 8y + 4 = 0$ or $x^2 + y^2 - 6x + 4y = 0$

Sol 24: Equation of circle touching other. circle is at point is $s + \lambda(L) = 0$

Where L is equation of tangent at the point

$x^2 + y^2 + 4x - 6y - 3 + \lambda(2x + 3y$

$+ 2(x + 2) - 3(y + 3) - 3) = 0$

It passes through (1, 1)

$\therefore \lambda = \frac{-(1+1+4-6-3)}{(2+3+6-12-3)} = \frac{3}{-4} = \frac{-3}{4}$

$\therefore$ Equation of circle is

$4x^2 + 4y^2 + 16x - 24y - 12 - 3(4x - 8) = 0$

$4x^2 + 4y^2 + 4x - 24y + 12 = 0$

$x^2 + y^2 + x - 6y + 3 = 0$

Exercise 2

Single Correct Choice Type

Sol 1: (A) Since BAC = 90°

locus of A is the circle with (3, 0), (–3, 0) as diameter

Let A = (h, – k)

$(h - 3)(h + 3) + k^2 = 0$

Now, centroid

$C(x, y) = \left(\frac{h+3-3}{3}, \frac{k+0+0}{3}\right)$

Substituting h, k in terms of (x, y)

$(3x - 3)(3x + 3) + (3y)^2 = 0$

$x^2 + y^2 = 1$ is the equation of centroid

Sol 2: (C) $|y| = x + 1$ & $(x - 1)^2 + y^2 = 4$

Substituting value of | y |

$(x - 1)^2 + (x + 1)^2 = 4$

$x^2 = 1$

$x = \pm 1$

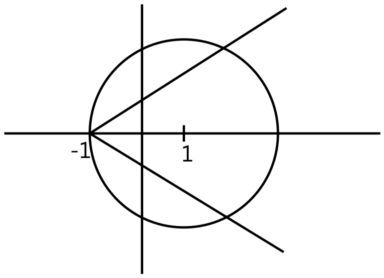
For x = –1; y = 0

For x = +1; | y | = 2 $\therefore$ y ± 2

$\therefore$ Three possible solutions are possible

Alternate method

Plotting the graph of $|y| = x + 1$ and $(x - 1)^2 + y^2 = 4$



We can directly see that three possible intersection are possible

Sol 3: (B) Line 1 passes through (3, 1) and

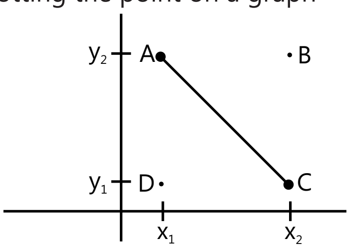
Line 2 passes through (1, 3)

Lines L_1 and L_2 are $\perp$ $\therefore$ locus of point of intersection is a circle with (3, 1) & (1,3) as ends of diameter

Locus of points is $(x - 3)(x - 1) + (y - 1)(y - 3) = 0$

$\therefore x^2 + y^2 - 4x - 4y + 6 = 0$

Sol 4: (B) Plotting the point on a graph



It is not necessary that

$|x_2 - x_1| = |y_2 - y_1|$

With (x_2, y_1) & (x_1, y_2) as ends of diameter $\angle ABC = 90°$ and $\angle ADC = 90°$

$\therefore$ ABCD are concyclic

Sol 5: (D) Let A = (0, 6), B = (5, 5) & C = (–1, 1)

Slope of AB = $\frac{-1}{5}$ & $m_{AC} = 5$

$\therefore$ AB $\perp$ AC

Circumcentre is midpoint of BC

O = (2, 3)

And radius = $\frac{1}{2}\sqrt{6^2 + 4^2} = \sqrt{13}$

Now y = mx is tangent to the circle $\therefore \frac{3 - 2m}{\sqrt{1 + m^2}} = \sqrt{13}$

$4m^2 - 12m + 9 = 13m^2 + 13 \Rightarrow 9m^2 + 12m + 4 = 0$

$9m^2 + 6m + 6m + 4 = 0$

$(3m + 2)^2 = 0$

$m = -\frac{2}{3}$

$\therefore$ Equation of line is 3y + 2x = 0

Sol 6: (A) The circumcenter of triangle A,B,C is (0, 0)

Let c $\equiv$ (h , k)

And centroid (c_1) is $\left(\frac{1+h}{3}, \frac{1+k}{3}\right)$

Let the orthocentre be (x, y)

The centroid divides O and C in ratio 2 : 1

$\therefore \left(\frac{1+h}{3}, \frac{1+k}{3}\right) = \left(\frac{x}{3}, \frac{y}{3}\right)$

$\therefore h = (x - 1)$ and $k = (y - 1)$

$(x - 1)^2 + (y - 1)^2 = 1$

$\therefore x^2 + y^2 - 2x - 2y + 1 = 0$

Sol 7: (D) Centre of circle is (– 8, –6)

Equation of line is y = 2x + 5

$\therefore$ Q is the foot of perpendicular of (– 8, –6) on 2x – y + 5 = 0

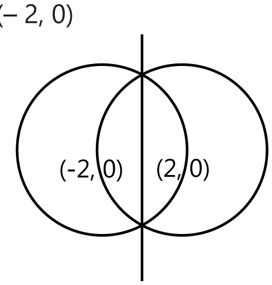
$\therefore \frac{x - (-8)}{2} = \frac{y - (-6)}{-1} = \frac{-(-5)}{5}$

$\therefore$ x = – 6 & y = – 7

$\therefore Q \equiv (-6, -7)$

Sol 8: (A) Centre of C_1 = (2, 0) R_1 = 4 & R_2 = 4

Centre of C_2 = (– 2, 0)



$\therefore$ The other 2 points of rhombus lie on y axis put in x = 0 we get

Y = ± 2 $\sqrt{3}$

$\therefore$ Length of 1st diagonal is (2 – (–2) = 4 and length of 2nd diagonal = $4\sqrt{3}$

Area of rhombus = $\frac{1}{2}ab = \frac{1}{2}\times 16\sqrt{3} = 8\sqrt{3}$ sq. units

Sol 9: (A) From (3, 4) chords are drawn to

$x^2 + y^2 - 4x = 0$

Let mid points of chord be (b, h)

$\therefore h^2 + k^2 - 4h = xh + yk - 2(h + x)$

Now (3, 4) pass through these chords

$\therefore h^2 + k^2 - 4h = 3h + 4k - 2(h + 3)$

$\therefore$ Locus of mid-point is $x^2 + y^2 - 5x - 4y + 6 = 0$

Sol 10: (B) Let p = (x, y)

$$(x, y) = \left(\frac{20\cos\theta+15}{5}, \frac{20\sin\theta+15}{5}\right)$$

$$\cos\theta = \frac{x-1}{4} \ \& \ \sin\theta = \frac{y-1}{4}$$

$(x - 1)^2 + (y - 1)^2 = 16$

This is a circle.

Sol 11: (B) (3, 4) & (–1, –2) are ends of diameter

$(x - 3)(x + 1) + (y - 4)(y + 2) = 0$

$x^2 + y^2 - 2x - 2y - 11 = 0$

Sol 12: (A) Shortest distance from line to circle

= $\perp$ distance – radius

Centre of circle $\equiv$ (3, –4) & radius = 5

$$\therefore \perp \text{ distance} = \left|\frac{9-16-25}{\sqrt{25}}\right| = \frac{32}{5}$$

$$\therefore \text{shortest distance} = \frac{32}{5} - 5 = \frac{7}{5}$$

Sol 13: (A) Slope of the line is 1

$\therefore y = x + c$

The two circle are

$s_1 \equiv x^2 + y^2 = 4$

$c_1 = (0, 0)$ & R = 2

$s_2 \equiv x^2 + y^2 - 10x - 14y + 65 = 0$

$c_2 = (5, 7)$ & R = 3

Length intercepted = $2\sqrt{R^2 - (\perp \text{distance})^2}$

$$\therefore \lambda_1 = 2\sqrt{2^2 - \left(\frac{O-O+C}{\sqrt{2}}\right)^2} = 2\sqrt{2^2 - \frac{C^2}{2}}$$

$$\lambda_2 = 2\sqrt{(3)^2 - \frac{(5-7+C)^2}{2}}$$

$\lambda_1 = \lambda_2$

$$\therefore 4 - \frac{C^2}{2} = 9 - \frac{(C-2)^2}{2}$$

$\therefore C^2 - 4C + 4 - C^2 = 10$

$$C = -\frac{3}{2}$$

Line is $y = x - \frac{3}{2}$

$2x - 2y - 3 = 0$

Sol 14: (D) Equation of circle is $x^2 + y^2 = r^2$

Let P $\equiv$ (a, b)

Let the midpoint of a point (h, k) on circle & P(a, b) be M(x, y)

$$(x, y) = \left(\frac{a+h}{2}, \frac{b+k}{2}\right)$$

h = 2x – a ; k = 2y – b

$(2x - a)^2 + (2y - b)^2 = r^2$ is locus of M

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \left(\frac{r}{2}\right)^2$$

Multiple Correct Choice Type

Sol 15: (C, D) Let h, k be the point of intersection

$\therefore$ Slope of lines is $\left(\frac{k}{h-1}\right)$ and $\left(\frac{k}{h+1}\right)$

For point (1, 0) and (–1, 0)

And $\tan(\theta - \theta_1) = \frac{\tan\theta - \tan\theta_1}{1+\tan\theta\tan\theta_1}$

$\therefore$ The angle between lines is either 45º or 135º

$\theta - \theta_1$ = 45º or 135º

$$\pm 1 = \frac{\frac{k}{h-1} - \frac{k}{h+1}}{1+\frac{k^2}{h^2-1}}$$

$$\pm 1 = \frac{2k}{h^2+k^2-1}$$

$\therefore h^2 + k^2 - 2k - 1 = 0$ $\quad C \equiv (0, 1)\ R = \sqrt{2}$

$\therefore h^2 + k^2 + 2k - 1 = 0$ $\quad C \equiv (0, -1)\ R = \sqrt{2}$

Sol 16: (B, C, D) $s_1 \equiv x^2 + y^2 + 2x + 4y + 1 = 0$

$s_2 \equiv x^2 + y^2 - 4x + 3 = 0$

$s_3 \equiv x^2 + y^2 + 6y + 5 = 0$

Radical axes of s_1 and s_2 is

$6x + 4y - 2 = 0$

$3x + 2y - 1 = 0$

Radical axes of s_3 and s_2 is

$6y + 4x + 2 = 0$

$3y + 2x + 1 = 0$

$5x + 3y = 0$

$x = 1\ y = -1$

(1, -1) is the radical centre

It is a known property that circle which is orthogonal to 3 circle has its center equal to radical center & radius = length of tangent from radical center to any circles.

Radices = $\sqrt{1+1+2-4+1} = 1$

Equation of orthogonal circle is $(x-1)^2 + (y+1)^2 = 1$

This circle touches both x & y axis.

Its x & y-intercept are 1

Sol 17: (B, C) $c_1 \equiv x^2 + y^2 - 4x + 6y + 8 = 0$

$c_2 \equiv x^2 + y^2 - 10x - 6y + 14 = 0$

Centre of $c_1 \equiv (2, -3)$

Centre of $c_2 \equiv (5, 3)$

$r_1 = \sqrt{4+9-8} = \sqrt{5}$

$r_2 = \sqrt{25+9-14} = 2\sqrt{5}$

$c_1c_2 = r_1 + r_2$

$c_1c_2 = \sqrt{(5-3)^2 + (6)^2} = 3\sqrt{5}$

$\therefore$ c_1 & c_2 touch each other

$\therefore$ Radical axis is the common tangent and the mid-point of c_1c_2 doesn't lie on radical axis as their radius are not the same.

Sol 18: (B, D) A = (–1, 1); B = (0, 6); C = (5, 5)

$AB \perp BC$

$\therefore$The circle passing through ABC will have AC as a diameter

$S : (x + 1)(x - 5) + (y - 1)(y - 5) = 0$

$\therefore x^2 + y^2 - 4x - 6y = 0$

Center c = (2, 3) ; r = $\sqrt{13}$

The line joining origin to center is $y = \frac{3}{2}x$

$\therefore 3x - 2y = 0$

The points will lie on the line $\perp$ to $3x - 2y = 0$ & passing through (2, 3) at a distance of r from (2, 3)

$L : y - 3 = \frac{-2}{3}(x - 2)$ $\tan\theta = \frac{-2}{3}$

$2x + 3y - 13 = 0$

Let points be (h, k)

When θ is in 2[nd] quadrant

$\sin\theta > 0$ & $\cos\theta < 0$

$h = a + r\cos\theta$; $k = a + r\sin\theta$

$\therefore h = 2 + \sqrt{13} \times \frac{-3}{\sqrt{13}}$

$k = 3 + \sqrt{13} \times \frac{2}{\sqrt{13}}$

$\therefore P_1 = (-1, 5)$

When q lies in 4[th] quadrant

$\sin\theta < 0$ & $\cos\theta > 0$

$h = 2 + \frac{3}{\sqrt{13}} \times \sqrt{13}$

$k = 3 + \left(\frac{-2}{\sqrt{13}} \times \sqrt{13}\right)$

$\therefore P_2 = (5, 1)$

Sol 19: (A, C, D) $s_1 : x^2 + y^2 + 2x + 4y - 20 = 0$

$s_2 \equiv x^2 + y^2 + 6x - 8y + 10 = 0$

$c_1 = (-1, -2)$ & $c_2 = (-3, 4)$

$r_1 = \sqrt{1^2 + 2^2 + 20} = 5$

$r_2 = \sqrt{3^2 + 4^2 - 10} = \sqrt{15}$

$c_1c_2 = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$

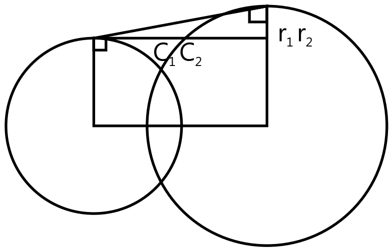
$c_1c_2 = r_1 + r_2$

and $c_1c_2 > |r_1 - r_2|$

$\therefore$ The two circles intersect each other at 2 points

$2gg_1 + 2ff_1 = 2 \times 3 + 4x - 4 = -10 = c + c_1$

The 2 circle are orthogonal



Length of common tangents

$= \sqrt{(c_1c_2)^2 - (r_2 - r_1)^2} = \sqrt{40 - (5 - \sqrt{15})^2}$

$= \sqrt{10\sqrt{15}} = 5(12/5)^4$

The equation of common chord is $s_1 - s_2$

$\Rightarrow 4x - 12y + 30 = 0$

$\Rightarrow 2x - 6y + 15 = 0$

Perpendicular from c_1 on this $\Rightarrow \dfrac{-2+12+15}{\sqrt{40}} = \dfrac{25}{\sqrt{40}}$

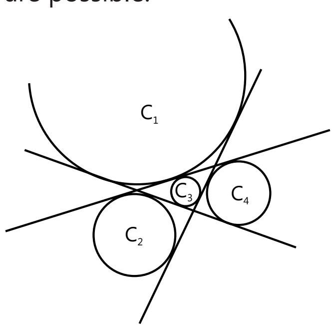
Length of common chord $= 2\sqrt{r^2 - a^2}$

$= 2\sqrt{25 - \left(\dfrac{25}{\sqrt{40}}\right)^2} = 2\sqrt{25 - \dfrac{625}{40}}$

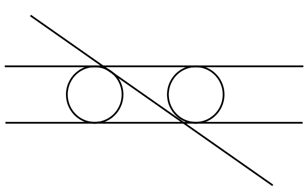
$= 2\sqrt{\dfrac{75}{8}} = \dfrac{10}{2}\sqrt{\dfrac{3}{2}} = 5\sqrt{\dfrac{3}{2}}$

Sol 20: (A, C, D) Consider 2 lines not parallel to one another and when the third line passes through intersection of both lines, no circle is possible.

When the third line doesnot pass through point of intersection of the lines & is not parallel to either of them 4 circle are possible.



When the 3rd line is parallel to one of the line then



2 circle are possible

When all 3 lines are parallel no circles are possible

Sol 21: (A, B, D) $c_1 = (x + 7)^2 + (y - 2)^2 = 25$

$\therefore r_1 = 5$

c_2 is director circle of c_1

$\therefore r_2 = 5\sqrt{2}$

And c_3 director circle of c_2

$\therefore r_3 = 5\sqrt{2} \times \sqrt{2} = 10$

Area enclosed by $c_3 = \pi r^2 = 100\pi$

Area enclosed of $c_2 = \pi \times (\sqrt{2}\,r)^2 = 2\pi r^2$

= 2 times area enclosed by c_1

Sol 22: (B, C) $S_1 \equiv x^2 + y^2 - 2x - 4y + 1 = 0$ $r_1 = 2$

$G \equiv (1, 2), r_1 = 2$

$S_2 \equiv x^2 + y^2 + 4x + 4y - 1 = 0$

$C_2 \equiv (-2, -2), r_2 = 3$

$C_1C_2 = \sqrt{3^2 + 4^2} = 5$

The two circle touch each other externally and common tangent is $S_2 - S_1 = 0$

$6x + 8y - 2 = 0$

$3x + 4y - 1 = 0$

Sol 23: (A, C, D) $S_1 \equiv x^2 + y^2 - 6x - 6y + 9 = 0$

$S_2 \equiv x^2 + y^2 + 6x + 6y + 9 = 0$

$C_1 = (-g, -t) = (3, 3)$

$r_1 = \sqrt{3^2 + 3^2 - 9} = 3$

and $C_2 = (-3, -3)$

$r_2 = \sqrt{3^2 + 3^2 - 9} = 3$

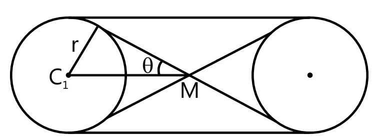
$C_1C_2 = \sqrt{6^2 + 6^2} = 6\sqrt{2}$

$r_1 + r_2 = 6$

They do not intersect with each other

Since their radius are same

$\therefore$ External direct common tangents are parallel



Also, the point of intersection of transverse common tangents is midpoint of C_1 and C_2 (same radii)

M = (0, 0)

$\sin\theta = \frac{r}{MC_1} = \frac{3}{\sqrt{3^2+3^2}} = \frac{1}{\sqrt{2}}$

$\theta = 45^\circ$

Angle between tangents = $2\theta = 90^\circ$

Sol 24: (B, C) $S_1 \equiv x^2 + y^2 + px + py - 7 = 0$

$S_2 \equiv x^2 + y^2 - 10x + 2py + 1 = 0$

S_1 & S_2 are orthogonal

$\therefore 2gg_1 + 2ff_1 = c + c_1$

$\Rightarrow p(-5) + p \,.\, p = -6$

$\Rightarrow\ p^2 - 5p + 6 = 0$

$\Rightarrow$ P =2 or p = 3

Sol 25: (A, B, D) (A) Two circles having the same center. Have infinitely many common normal.

(B) Radical axis is always perpendicular to the line joining center but it does not necessarily bisect the line joining the centres. It bisects only when $r_1 = r_2$

(C) Let the centres of the two circles be C_1 & C_2.

Consider a point O, on radical axis centres which lies on the line C_1C_2

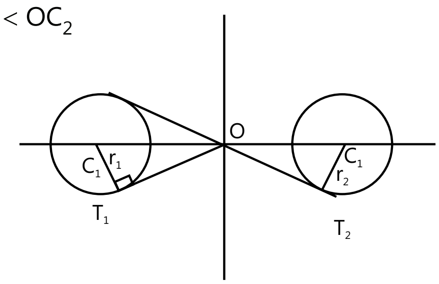
Now $OC_1^2 = r_1^2 + OT_1^2$

$OC_2^2 = r_2^2 + OT_2^2$

Since length of tangent is same

$\therefore\ OC_1^2 < OC_2^2$ if $r_1 < r_2$

$\Rightarrow OC_1 < OC_2$



(D) Consider two circles having same centre these circles donot have a radical axis

Assertion Reasoning Type

Sol 26: (C) $L : \overbrace{k(x-y-4)}^{L_1} + \overbrace{7x+y+20}^{L_2} = 0$

L are the lines passing through intersection of L_1 & L_2

Point of intersection is (– 2, – 6)

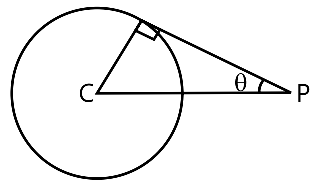
Which is center of circle c

Every line L is normal to circle

Statement-I is true & statement-2 is false

Sol 27: (A) Length of tangent from (13, 6)

$= \sqrt{13^2 + 6^2 - 13\times 6 + 8\times 6 - 75} = 10$



$\therefore$ Radius of circle $= \sqrt{3^2 + (-4)^2 + 75} = 10$

$\therefore \tan\theta = 1$

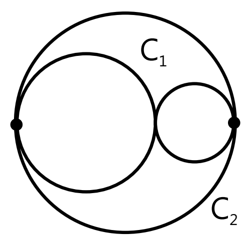
$\therefore \theta = 45^\circ$

Angle between tangents = $2\theta = 2 \times 45 = 90^\circ$

Director circle of a circle S_1 is such that the angle between the tangents drawn from any point on director circle to S_1 is 90°

Sol 28: (D) (1, 5) lies outside the circle

as 1 + 25 – 2 – 7 = 17 > 0



$\therefore$ Two circles shown C_1, C_2 are possible

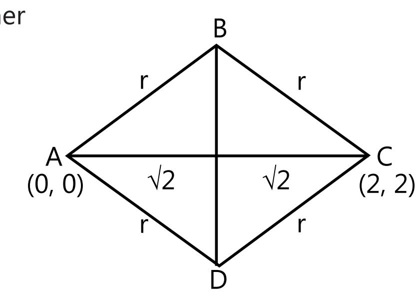
$\therefore$ Statement-I is false

Sol 29: (A) Since x + y – 2 = 0 is $\perp$ bisector of C_1C_2

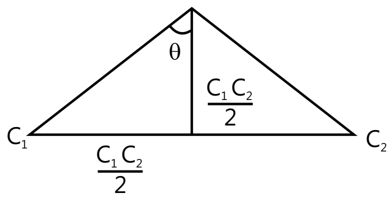
Radius of both the circles is same

Since length of common chord = $2\sqrt{2}$

ABCD is a square since diagonals are equal & $\perp$ to each other



When their centres are mirror image of each other then the common chord bisects C_1C_2 and $\frac{1}{2}\times$ length of common chord = $\frac{1}{2}c_1c_2$



$\tan\theta = 1$

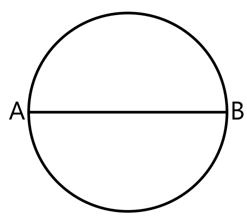
$\theta = 45^o$

The circles are orthogonal

When the centres are mirror image & length of chord = distance between centres then the two circles are orthogonal. The inverse is not true

$\therefore$ Statement-II is wrong

Sol 30: (A) Let AB = diameter



The circle with AB as diameter is

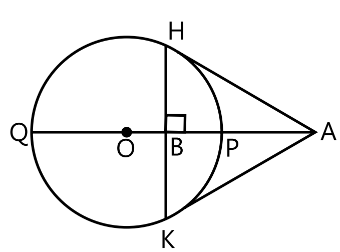
$(x - x_1)(x - x_2) + (y - y_1).(y - y_2) = 0$

when C is obtuse, then C lies inside the circle

$D(x_3, y_3) < 0$

(Power of a point inside a circle < 0)

Sol 31:



Since KPHQ are concyclic

$\therefore$ PB × BQ = HB × BK = $(HB)^2$

$\therefore$ (AB – AP) (AQ – AB) = $(HB)^2$

Also AH^2 = AP × AQ (from property of tangents)

$AH^2 - HB^2 = AP \times AQ - [AB \times AP$

$+ AB \times AQ - AB^2 - APAQ]$

$AB^2 = AP \times AQ - [AB(AP + AQ) - AB^2 - APAQ]$

$\therefore AB = \dfrac{2AP \times AQ}{AP + AQ}$

$\therefore$ Statement-I is true

Statement-II: $AK^2 = AB \times AO$ & $AK^2 = AP \times AQ$

$\therefore AB \times \dfrac{(AP + AQ)}{2} = AP \times AQ$

$\therefore AB = \dfrac{2AP \times AQ}{(AP + AQ)}$

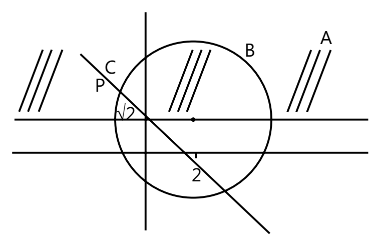
Comprehension Type

Paragraph 1: (32-34)

Sol 32: (B) A : {(x, y) : y ≥ 1 }

B : $\{(x, y) : x^2 + y^2 - 4x - 2y - 4 = 0\}$

C : $\{(x, y) : x + y = \sqrt{2}$



There is only one point P of intersection of region A, B, C

Sol 33: (C) B : $x^2 + y^2 - 4x - 2y - 4 = 0$

$\Rightarrow 2x^2 + 2y^2 - 8x - 4y - 8 = 0$

$\Rightarrow (x - 5)^2 + (x + 1)^2 + (y - 1)^2 + (y - 1)^2 - 36 = 0$

$\therefore f(x) = 36$

Sol 34: (C) S is director circle of B

$\therefore B : (x - 2)^2 + (y - 1)^2 = 9$

$s : (x - 2)^2 + (y - 1)^2 = 18$

Arc of B = 9π

Arc of s = 18π

Area of S-Area of B = 9π

Paragraph 2: (35-36)

Sol 35: (D) Let m be slope of tangents

$\therefore$ (y – 2) = m(x – 4) are equation of tangent

$s = x^2 + y^2 = 4$

For tangents $c^2 = a^2(1 + m^2)$

$\therefore (2 - 4m)^2 = 4(1 + m^2)$

$12m^2 - 16m = 0$

$4m(3m - 4) = 0$

$m = 0$ or $m = \frac{4}{3} \Rightarrow \tan\theta = \frac{4}{3}$

$\theta \in (45^o, 60^o)$ Ans.(D)

Sol 36: (B) the tangents are

y = 2 & 4x – 3y – 10 = 0

$\therefore$ Intercepts made on x axis by 2nd tangent $= \frac{10}{4} = \frac{5}{2}$

Paragraph 3: (37-39)

Sol 37: (D) $s : x^2 + y^2 - 4x - 1 = 0$

L : y = 3x – 1

Centre of circle = (2, 0)

Radius = $\sqrt{5}$

Length of chord AB

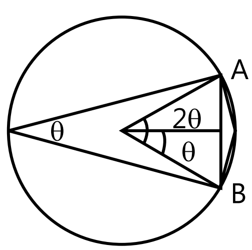
$= 2\sqrt{r^2 - (\text{perpendicular distance from centre})^2}$

Perpendicular distance from centre

$= \frac{6-1}{\sqrt{10}} = \sqrt{\frac{5}{2}}$

$AB = 2\sqrt{5 - \frac{5}{2}} = 2\sqrt{\frac{5}{2}} = \sqrt{10}$

Sol 38: (A)



Angle subtends at minor arc = 180 – angle at major arc

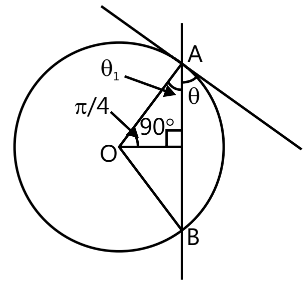
$\tan\theta = \frac{1}{2}\frac{\ell_{AB}}{\pm\text{distance}}$

$= \frac{1}{2} \times \frac{\sqrt{10}}{\sqrt{5/2}} = 1$

$\theta = 45^o$

Angle at minor arc $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Sol 39: (C)



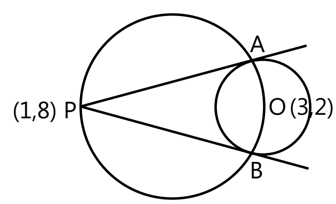
$\theta_1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

& $\theta + \theta_1 = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

Previous Years' Questions

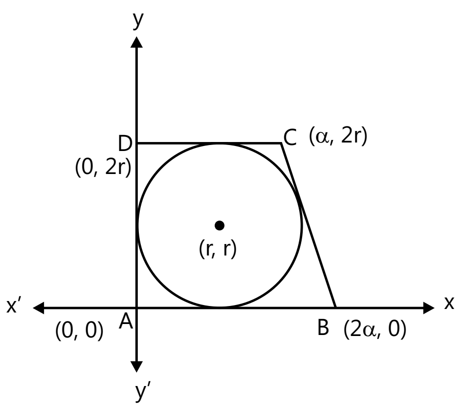
Sol 1: (B) For required circle, P(1, 8) and O(3,2) will be the end point of its diameter.



$\therefore (x-1)(x-3) + (y-8)(y-2) = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 10y + 19 = 0$

Sol 2: (B) $18 = \frac{1}{2}(3\alpha)(2r) \Rightarrow \alpha r = 6$

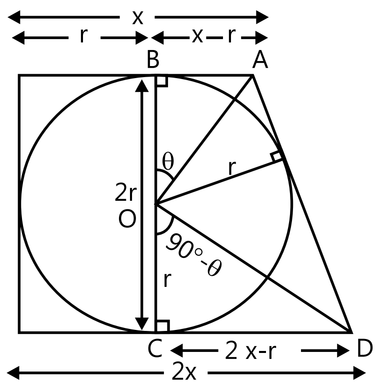


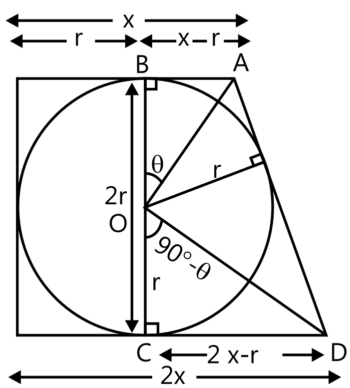
Line, $y = -\frac{2r}{\alpha}(x - 2\alpha)$ is tangent to circle

$(x-r)^2 + (y-r)^2 = r^2$

$2\alpha = 3r$ and $\alpha r = 6$

$r = 2$





Alternate solution

$\frac{1}{2}(x + 2x) \times 2r = 18$

$xr = 6$ ...(i)

In ΔAOB, $\tan\theta = \frac{x-r}{r}$

and in ΔDOC

$\tan(90^o - \theta) = \frac{2x - r}{r}$

$\therefore \frac{x-r}{r} = \frac{r}{2x-r}$

$\Rightarrow \quad x(2x - 3r) = 0$

$\Rightarrow \quad x = \frac{3r}{2}$ ....(ii)

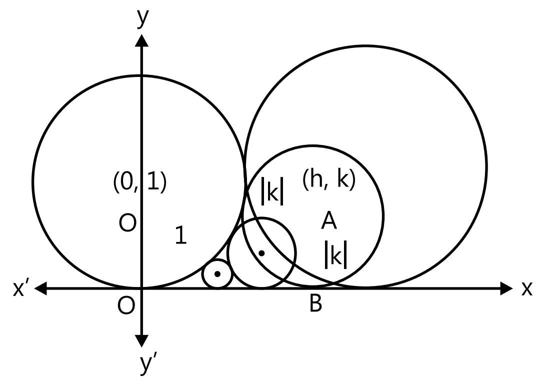
From Eqs. (i) and (ii) we get

r = 2

Sol 3: (A) Let the locus of centre of circle be (h, k) touching

$(y-1)^2 + x^2 = 1$ and x-axis shown as

Clearly, from figure,



Distance between O and A is always 1+|k|,

ie, $\sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|,$

$\Rightarrow \quad h^2 + k^2 - 2k + 1$

$= 1 + k^2 + 2|k|$

$\Rightarrow \quad h^2 = 2|k| + 2k$

$\Rightarrow \quad x^2 = 2|y| + 2y$

where $|y| = \begin{cases} y, & y \geq 0 \\ -y, & y < 0 \end{cases}$

$\therefore \quad x^2 = 2y + 2y, y \geq 0$

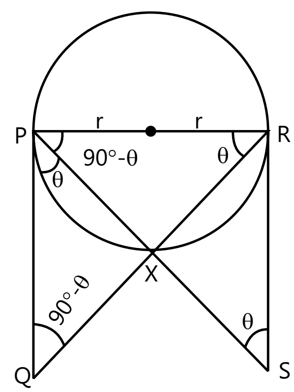
and $\quad x^2 = 2y + 2y, y < 0$

$\Rightarrow \quad x^2 = 4y \quad$ when $y \geq 0$

and $\quad x^2 = 0 \quad$ when $y < 0$

$\therefore \quad \{(x, y) : x^2 = 4y, \text{ when } y \geq 0\} \cup \{(0, y) : y < 0\}$

Sol 4: (A) From figure it is clear that ΔPRQ and ΔRSP are similar.



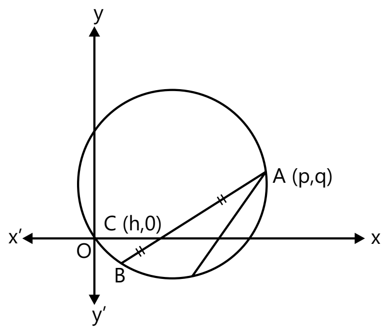
$\therefore \quad \dfrac{PR}{RS} = \dfrac{PQ}{RP}$

$\Rightarrow \quad PR^2 = PQ \cdot RS$

$\Rightarrow \quad PR = \sqrt{PQ.RS}$

$\Rightarrow \quad 2r = \sqrt{PQ \cdot RS}$

Sol 5: (B) Choosing OA as x-axis, A=(r, 0), B=(0, r) and any point P on the circle is $(r\cos\theta, r\sin\theta)$. If (x, y) is the centroid of ΔPAB, then



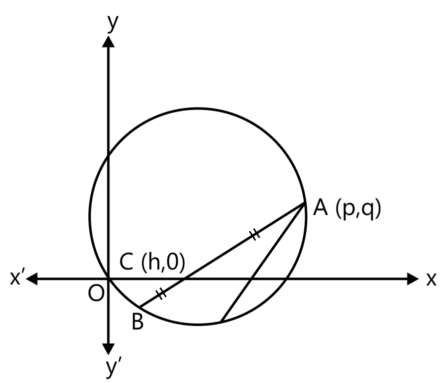
$$3x = r\cos\theta + r + 0$$

and $\quad 3y = r\sin\theta + 0 + r$

$\therefore \quad (3x - r)^2 + (3y - r)^2 = r^2$

Hence, locus of P is a circle.

Sol 6: (D) From equation of circle it is clear that circle passes through origin. Let AB is chord of the circle.



$A \equiv (p,q) \cdot C$ is mid point and coordinate of C is (h, 0)

Then coordinates of B are (-p + 2h, -q) and B lies on the circle $x^2 + y^2 = px + qy$. we have

$$(-p + 2h)^2 + (-q)^2 = p(-p + 2h) + q(-q)$$

$\Rightarrow \quad p^2 + 4h^2 - 4ph + q^2 = -p^2 + 2ph - q^2$

$\Rightarrow \quad 2p^2 + 2q^2 - 6ph + 4h^2 = 0$

$\Rightarrow \quad 2h^2 - 3ph + p^2 + q^2 = 0 \qquad$ ....(i)

There are given two distinct chords which are bisected at x-axis then, there will be two distinct values of h satisfying Eq. (i).

So, discriminant of this quadratic equation must be > 0.

$\Rightarrow \quad D > 0$

$\Rightarrow \quad (-3p)^2 - 4 \cdot 2(p^2 + q^2) > 0$

$\Rightarrow \quad 9p^2 - 8p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 > 8q^2$

Sol 7: Equation of given circle C is

$$(x - 3)^2 + (y + 5)^2 = 9 + 25 - 30$$

ie, $(x - 3)^2 + (y + 5)^2 = 2^2$

Centre = (3, -5)

If L_1 is diameter, then $2(3) + 3(-5) + p - 3 = 0 \Rightarrow p = 12$

$\therefore \; L_1$ is $2x + 3y + 9 = 0$

$\quad L_2$ is $2x + 3y + 15 = 0$

Distance of centre of circle from L_2 equals

$$\left|\frac{2(3) + 3(-5) + 15}{\sqrt{2^2 + 3^3}}\right| = \frac{6}{\sqrt{13}} < 2 \text{ (radius of circle)}$$

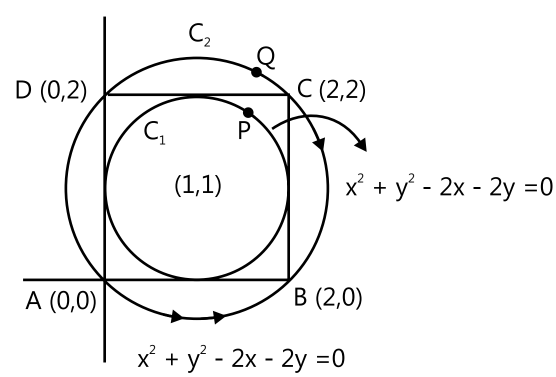
$\therefore \quad L_2$ is a chord of circle C.

Statement-II, false.

Sol 8: (A) Let the, equation of circles are

$$C_1 : (x - 1)^2 + (y - 1)^2 = (1)^2$$

and $\quad C_2 : (x - 1)^2 + (y - 1)^2 = (\sqrt{2})^2$



$\therefore$ Coordinates of $P(1 + \cos\theta, 1 + \sin\theta)$

and $\quad Q(1+\sqrt{2}\cos\theta, 1+\sqrt{2}\sin\theta)$

$\therefore \quad PA^2 + PB^2 + PC^2 + PD^2$

$$= \{(1+\cos\theta)^2 + 1 + \sin\theta)^2\} + \{(\cos\theta - 1)^2 + (1+\sin\theta)^2\} + \{(\cos\theta-1)^2 + (\sin\theta - 1)^2\} + \{(1+\cos\theta)^2 + (\sin\theta - 1)^2\}$$

$$= 12$$

Similarly, $QA^2 + QB^2 + QC^2 + QD^2 = 16$

$\therefore \quad \dfrac{\Sigma PA^2}{\Sigma QA^2} = \dfrac{12}{16} = 0.75$

Sol 9: (C) Let C be the centre of the required circle.

Now, draw a line parallel to L at a distance of r_1 (radius of C_1) from it.

Now, $CC_1 = AC$

$\Rightarrow$ C lies on a parabola.

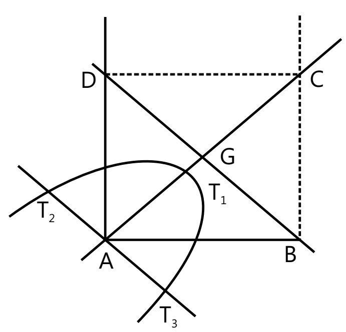
Sol 10: (C)

Since, $\quad AG = \sqrt{2}$

$\therefore \quad AT_1 = T_1G = \dfrac{1}{\sqrt{2}}$

As A is the focus, T_1 is the vertex and BD is the directrix of parabola.

Also, T_2T_3 is latus reetum.



$\therefore \quad T_2T_3 = 4 \cdot \dfrac{1}{\sqrt{2}}$

$\therefore$ Area of $\Delta T_1T_2T_3 = \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}} \times \dfrac{4}{\sqrt{2}} = 1$ sq unit

Sol 11: (D) Let centre of circle C be (h, k)

Then, $\quad \left|\dfrac{\sqrt{3}h + k - 6}{\sqrt{3+1}}\right| = 1$

$\Rightarrow \quad \sqrt{3}h + k - 6 = +2$

$\Rightarrow \quad \sqrt{3}h + k = 4 \quad$ ....(i)

(Rejecting '2' because origin and centre of C are on the same side of PQ).

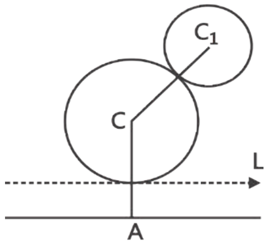
The point $(\sqrt{3}, 1)$ satisfies Eq. (i).

$\therefore \quad$ Equation of circle C is $(x-\sqrt{3})^2 + (y-1)^2 = 1.$

Sol 12: (A) Slope of line joining centre of circle to point D is

$$\tan\theta = \frac{\frac{3}{2} - 1}{\frac{3\sqrt{2}}{2} - \sqrt{3}} = \frac{1}{\sqrt{3}}$$

It makes an angle 30° with x-axis.



$\therefore$ Point E and F will make angle 150° and -90° with x-axis.

$\therefore$ E and F are given by

$$\frac{x-\sqrt{3}}{\cos 150^\circ} = \frac{y-1}{\sin 150^\circ} = 1$$

and $$\frac{x-\sqrt{3}}{\cos(-90^\circ)} = \frac{y-1}{\sin(-90^\circ)} = 1$$

$\therefore \quad E = \left(\dfrac{\sqrt{3}}{2}, \dfrac{3}{2}\right)$ and $F = (\sqrt{3}, 0)$

Sol 13: (D) Clearly, points E and F satisfy the equations given in option (d).

Sol 14:

$2x^2 + y^2 - 3xy = 0 \quad$ (given)

$\Rightarrow \quad 2x^2 - 2xy - xy + y^2 = 0$

$\Rightarrow \quad 2x(x-y) - y(x-y) = 0$

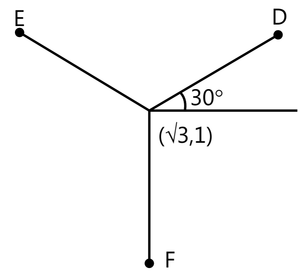
$\Rightarrow \quad (2x-y)(x-y) = 0$

$\Rightarrow \quad y = 2x, y = x$ are the equations of straight lines passing through origin.

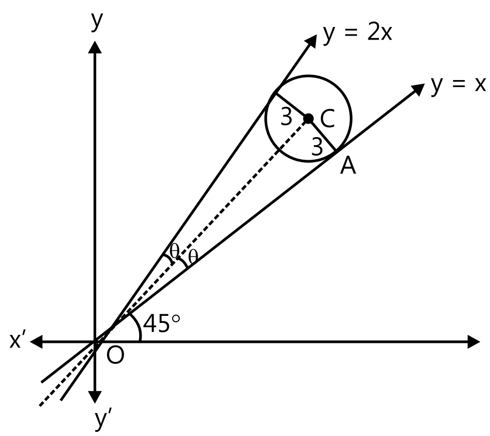
Now, let the angle between the lines be 2θ and the line y = x

Makes angle of 45° with x-axis.

Therefore, $\tan(45^\circ + 2\theta) = 2$ (slope of the line y =2x)



$$\Rightarrow \quad \frac{\tan 45^{\circ} + \tan 2\theta}{1 - \tan 45^{\circ} \times \tan 2\theta} = 2$$



$$\Rightarrow \quad \frac{1 + \tan 2\theta}{1 - \tan 2\theta} = 2$$

$$\Rightarrow \quad \frac{(1 + \tan 2\theta) - (1 - \tan 2\theta)}{(1 + \tan 2\theta) + (1 - \tan 2\theta)} = \frac{2 - 1}{(2 + 1)} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{2\tan 2\theta}{2} = \frac{1}{3}$$

$$\Rightarrow \tan 2\theta = \frac{1}{3}$$

$$\Rightarrow \quad \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{1}{3}$$

$$\Rightarrow \quad (2\tan\theta) \cdot 3 = 1 - \tan^2\theta$$

$$\Rightarrow \quad \tan^2\theta + 6\tan\theta - 1 = 0$$

$$\Rightarrow \quad \tan\theta = \frac{-6 \pm \sqrt{36 + 4 \times 1 \times 1}}{2}$$

$$\Rightarrow \quad \tan\theta = \frac{-6 \pm \sqrt{40}}{2}$$

$$\Rightarrow \quad \tan\theta = -3 \pm \sqrt{10}$$

$$\Rightarrow \quad \tan\theta = -3 + \sqrt{10} \qquad \left(\because 0 < \theta < \frac{\pi}{4}\right)$$

Again, in ΔOCA

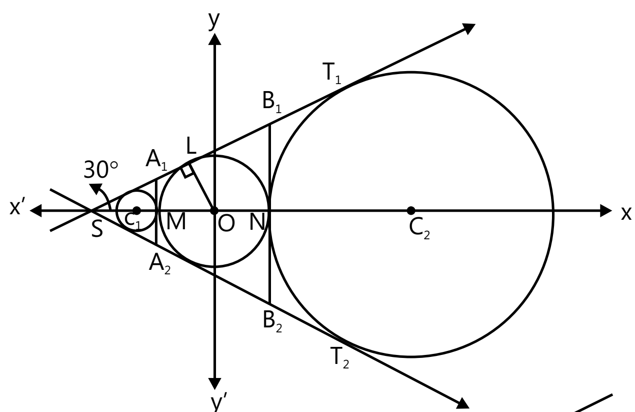
$$\tan\theta = \frac{3}{OA}, OA = \frac{3}{\tan\theta}$$

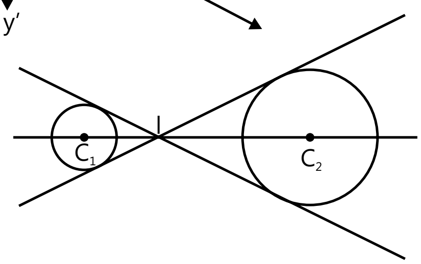
$$= \frac{3}{(-3 + \sqrt{10})}$$

$$\therefore \quad = \frac{3(3 + \sqrt{10})}{(-3 + \sqrt{10})(3 + \sqrt{10})}$$

$$= \frac{3(3 + \sqrt{10})}{(10 - 9)} = 3(3 + \sqrt{10})$$

Sol 15:





From figure it is clear that, triangle OLS is a right triangle with right angle at L.

Also, OL = 1 and OS = 2

$\therefore \quad \sin(\angle LSO) = \frac{1}{2} \Rightarrow \angle LSO = 30^{\circ}$

Since, $SA_1 = SA_2$, ΔSA_1A_2 is an equilateral triangle.

The circle with centre at C_1 is a circle inscribed in the ΔSA_1A_2. Therefore, centre C_1 is centroid of ΔSA_1A_2. This, C_1 divides SM in the ratio 2:1. Therefore, coordinates of C_1 are (-4/3,0) and its radius C_1 M=1/3

$\therefore$ Its equation is $(x + 4/3)^2 + y^2 = (1/3)^2$ ...(i)

The other circle touches the equilateral triangle SB_1B_2

Externally. Its radius r is given by $r = \frac{\Delta}{s - a}$,

where $B_1B_2 = a$. But $\Delta = \frac{1}{2}(a)(SN) = \frac{3}{2}a$

and $\quad s - a = \frac{3}{2}a - a = \frac{a}{2}$

Thus, $\quad r = 3$

$\Rightarrow$ Coordinates of C_2 are (4,0)

$\therefore$ Equation of circle with centre at C_2 is

$(x-4)^2 + y^2 = 3^2$ ...(ii)

Equations of common tangents to circle (i) and circle C are

$x = -1$ and $y = \pm\frac{1}{\sqrt{3}}(x+2)$ [T_1 and T_2]

Equation of common tangents to circle (ii) and circle C are

$x = -1$ and $y = \pm\frac{1}{\sqrt{3}}(x+2)$ [T_1 and T_2]

Two tangents common to (i) and (ii) are T_1 and T_2 at O. To find the remaining two transverse tangents to (i) and (ii), we find a point I which divides the joint of C_1C_2 in the ratio $r_1 : r_2 = 1/3 : 3 = 1 : 9$

Therefore, coordinates of I are (-4/5,0)

Equation of any line through I is y = m (x+4/5). It will touch (i) if

$$\frac{\left|m\left(\frac{-4}{3}+\frac{4}{5}\right)-0\right|}{\sqrt{1+m^2}} = \frac{1}{3}$$

$\Rightarrow$ $\left|-\frac{8m}{15}\right| = \frac{1}{3}\sqrt{1+m^2}$

$\Rightarrow$ $64m^2 = 25(1+m^2)$

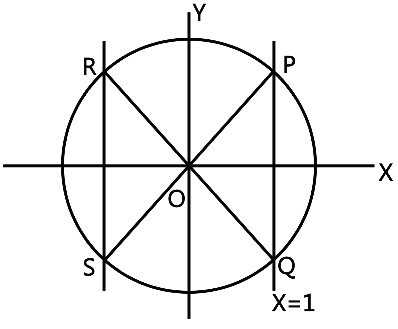
$\Rightarrow$ $39m^2 = 25$

$\Rightarrow$ $m = \pm\frac{5}{\sqrt{39}}$

Therefore, these tangents are

$$y = \pm\frac{5}{\sqrt{39}}\left(x+\frac{4}{5}\right)$$

Sol 16: Let equation of Circle be $x^2 + y^2 = 4$ and parallel chords are $x = 1$ and -13



$P \equiv (1,13)$, $Q \equiv (1,-13)$

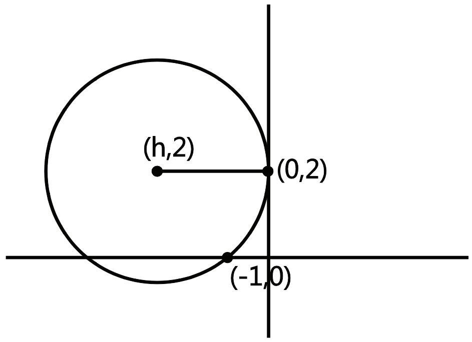
$R \equiv (-\sqrt{3},1), S \equiv (-\sqrt{3},-1)$

$\angle POQ = \frac{2\pi}{3} = \frac{\pi}{k}$

$\angle ROS = \frac{\pi}{3} = \frac{\pi}{k}$

$\Rightarrow k = 3$

Sol 17: (D)



$(x-h)^2 + (y-2)^2 = h^2$

Passes through $(-1,0)$, then

$(-1,-h)^2 + (0-2)^2 = h^2$

$(1+h)^2 - h^2 = -4$

$\Rightarrow (1+h-h)(1+h+h) = -4$

$\Rightarrow (1)(2h+1) = -4$

$h = -5/2$

Circle is

$$\left(x+\frac{5}{2}\right)^2 + (y-2)^2 = \left(\frac{5}{2}\right)^2$$

Only $(-4,0)$ satisfies the eq. of circle.

D is the Answer.

Sol 18: (D) Any tangent to circle $x^2 + y^2 = 4$ and $(x-3)^2 + y^2 = 1$, then

$$\frac{|3x_1 + 0\times y_1 - 4|}{\sqrt{x_1^2 + y_1^2}} = 1$$

$$\frac{|3x_1 - 4|}{y} = 1$$

$\Rightarrow |3x_1 - 4| = 2$

$\Rightarrow x_1 = 2, 2/3$

$\Rightarrow (x_1, y_1) \equiv (2,0)$ & $\left(\frac{2}{3}, \frac{4\sqrt{2}}{3}\right)$

Tangents

$2\,.\,x+0=4 \Rightarrow x=2$ and $\frac{2x}{3}+\frac{4\sqrt{2}}{3}=4$

$\Rightarrow x+2\sqrt{2}=6$

Sol 19: (A) The tangent to circle $x^2+y^2=4$ at $\left(\sqrt{3},1\right)4$

$PT \equiv \sqrt{3}x+y=4$

Eq. of L is $x-\sqrt{3}y=\lambda$

Circle $\left(x-3\right)^2+y^2=1$ is touching L, then

$$\frac{\left|3-\sqrt{3}\times 0-\lambda\right|}{\sqrt{1+3}}=1$$

$\left|3-\lambda\right|=2$

$\lambda=1,5$

Tangents $x-\sqrt{3}y=1$

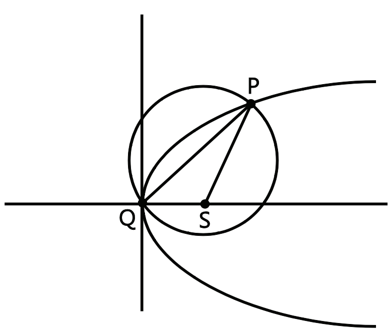
Sol 20: Let P be $(2t^2, 4t)$ lies on circle

$4t^4+16t^2-4t^2-16t=0$

$\Rightarrow t^4+4t^2-t^2-4t=0$

$\Rightarrow t\left(t-1\right)\left(t^2+t+4\right)=0$

$\Rightarrow t=0,1$



$P\equiv\left(2,4\right)$ $\qquad Q\equiv\left(0,0\right)$ $\qquad S\equiv\left(2,0\right)$

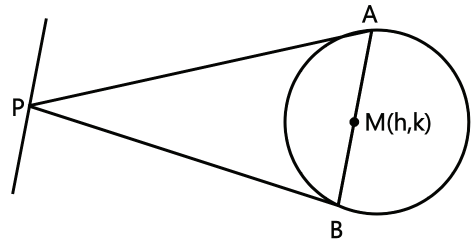
$\Delta=\frac{1}{2}\times 2\times 4=4$ sq units

Sol 21: (A) Let point P be $\left(t,\frac{4t-20}{5}\right)$

Eq. of chord of contact $xt+y\left(\frac{4t-20}{5}\right)=9$

$\left(5t\right)x+y\left(4t-20\right)=45$ ... (i)

If (h,k) mid-point, the eq. of chord of contact $T=S_1$



$xh+yk=h^2+k^2$ ... (ii)

(i) & (ii) are identical, then

$$\frac{h}{5f}=\frac{k}{4f-20}=\frac{h^2+k^2}{45}$$

$$t=\frac{9h}{h^2+k^2}$$

$$4t-20=\frac{45k}{h^2+k^2}$$

$$\Rightarrow \frac{9h\times 4}{h^2+k^2}-20=\frac{45k}{h^2+k^2}$$

$$\Rightarrow 36h-20\left(n^2+k^2\right)=45k$$

$$\Rightarrow 20\left(h^2+k^2\right)-36h+45k=0$$

$$\Rightarrow 20\left(x^2+y^2\right)-36x+45y=0$$

Sol 22: (C) Let circle touching x-axis be

$\left(x-\alpha\right)^2+\left(y-k\right)^2=k^2$ ... (i)

Also for y-axis intercepts

$\left(0-\alpha\right)^2=\left(y-k\right)^2=k^2$

$\Rightarrow \left(y-k\right)^2=k^2-\alpha^2$

$\Rightarrow y=k\pm\sqrt{k^2-\alpha^2}$

Intercept $=2\sqrt{k^2-\alpha^2}=2\sqrt{7}$

$\Rightarrow k^2=7+\alpha^2$

From (i) $\alpha=3$

$\Rightarrow k^2=7+9=16$

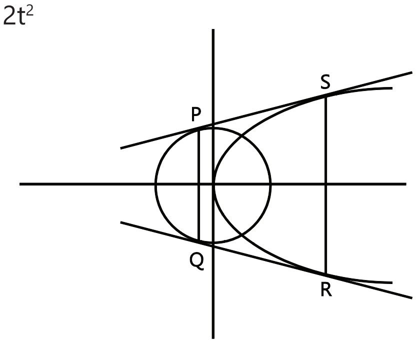
$\Rightarrow k=\pm 4$

Circle: $\left(x-3\right)^2+\left(y-4\right)^2=16$

$\left(x-3\right)^2+\left(y+4\right)^2=16$

Sol 23: (D) Let tangent to parabola $y^2=8x$

Be $ty=x+2t^2$



It is also tangent to circle, then

$$\frac{2t^2}{\sqrt{1+t^2}}=\sqrt{2}$$

$$\Rightarrow 4t^4=2\left(1+t^2\right)$$

$$\Rightarrow 2t^4-t^2-1=0$$

$$\Rightarrow \left(2t^2+1\right)\left(t^2-1\right)=0$$

$$\Rightarrow t=\pm 1$$

$$\Rightarrow S\equiv\left(2,4\right)\&R\equiv\left(2,-4\right)$$

$$\Rightarrow P\equiv\left(-1,1\right)\ \ \&\ \ Q\left(-1,1\right)$$

$$\text{Area}=\frac{1}{2}\left(2+B\right)\times 3=15\text{ sq units}$$

Sol 24: (B, C) Let circle be $x^2+y^2+2yx+2y+C=0$

Applying condition for orthogenality

$$2gx-1+2f\times 0=C+\left(-15\right)$$

$$\Rightarrow 2g+c=15\ \text{ and }\ 2g\times 0+2f\times 0=C-1$$

$$\Rightarrow C=1$$

$$\Rightarrow g=7$$

Also,

$$1+2f+C=0$$

$$\Rightarrow f=-1$$

$$\text{Centre}\equiv\left(-g,-f\right)\equiv\left(-7,1\right)$$

$$\text{Radius}=\sqrt{g^2+f^2-C}=\sqrt{49+1-1}=7$$

Hence, B and C are the correct options

Sol 25: (A, C, D) $(x-2)^2+(y-8)^2=4$

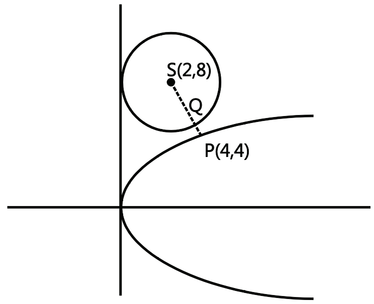
Shortest distance is measured along common normal

The equation of normal to parabola

$y=mx-2am-am^3 \Rightarrow y=mx-2m-m^3$

Passes through (2, 8), then

$8=2m-2m-m^3 \Rightarrow m=-2$



Normal

$y=-2x=12$

$$SP=\sqrt{\left(4-2\right)^2+\left(4-8\right)^2}=\sqrt{4+16}=2\sqrt{5}\text{ units}$$

Let SQ : QP = 1 : λ

$$\frac{1}{S\left(2,8\right)}.\frac{\lambda}{Q\left(h,k\right)P\left(4,4\right)}$$

$$Q\left(h,k\right)\equiv\left(\frac{4+2\lambda}{1+\lambda},\frac{8\lambda+y}{1+\lambda}\right)$$ lies on

Circle, then

$$\left(\frac{4+2\lambda}{1+\lambda}-2\right)^2+\left(\frac{8\lambda+4}{1+\lambda}-8\right)^2=4$$

$$\Rightarrow\left(\frac{2}{1+\lambda}\right)^2+\left(\frac{-4}{1+\lambda}\right)^2=4$$

$$\Rightarrow\frac{20}{\left(1+\lambda\right)^2}=4$$

$$\Rightarrow 1+\lambda=\sqrt{5}$$

$$\Rightarrow \lambda=\sqrt{5}-1$$

$$\frac{SQ}{QP}=\left(\sqrt{5}-1\right)$$

x – intercept of normal at P is 6 slope of tangent at Q is $\frac{1}{2}$

Sol 26: (C) For point of intersection

$2y+y^2=3$

$$\Rightarrow y^2+2y-3=0$$

$$\Rightarrow (y+3)(y-1)=0$$

$\Rightarrow y = 1, -3$

$\Rightarrow = \left(\sqrt{2}, 1\right)$

The eq. of tangent at $\left(1, \sqrt{2}\right)$

$\sqrt{2}x + y = 3$ (i)

Eqs. of circle C_2 and C_3

$C_2 \equiv x^2 + (y - y_2)^2 = 12$

$C_2 \equiv x^2 + (y - y_3)^2 = 12$

If line (i) touches circle, then

$$\left|\frac{\sqrt{2} \times 0 + y - 3}{\sqrt{2+1}}\right| = 2\sqrt{3}$$

$\Rightarrow |y - 3| = 6$

$\Rightarrow |y - 3| = \pm 6$

$\Rightarrow y = -3, 9$

$\Rightarrow y_2 = -3$ and $y_3 = 9$

$\Rightarrow$ Centres $Q_2 \equiv (0, -3)$

$Q_3 \equiv (0, 9)$

$\Rightarrow Q_2Q_3 = 12$

For point of contact R_2 and R_3

$R_2 \equiv \left(2\sqrt{2}, -1\right)$ and $R_3 \equiv \left(-2\sqrt{2}, 7\right)$

$$R_2R_3 = \sqrt{\left(4\sqrt{2}\right)^2 + (8)^2} = \sqrt{32 + 64} = \sqrt{96} = \sqrt{16 \times 6} = 4\sqrt{6}$$

$0(0,0)$, $R_2(\sqrt{2}, -1)$, $R_3(-2\sqrt{2}, 7)$

$$\text{Area of } \Delta OR_2R_3 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 2\sqrt{2} & -1 & 1 \\ -2\sqrt{2} & 7 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left(7 \times 2\sqrt{2} - 2\sqrt{2}\right) = \frac{1}{2} \times 6 \times 2\sqrt{2}$$

$= 6\sqrt{2}$ sq. units

Now

Area of ΔPQ_2Q_3

$$= \frac{1}{2}\begin{vmatrix} \sqrt{2} & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 9 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[\sqrt{2}(-3 - 9)\right] = 6\sqrt{2} \text{ sq units}$$

Sol 27: (A, C) Let point P be $(\cos\theta, \sin\theta)$, The tangent and normal are

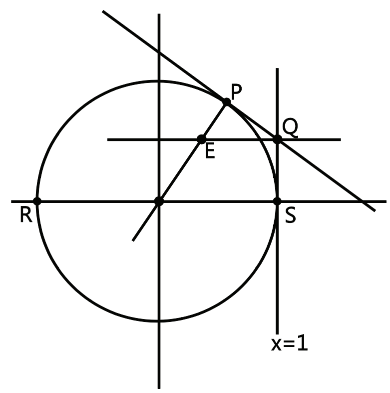
$x\cos\theta + y\sin\theta = 1 \quad x\sin\theta - y\cos\theta = 0$

$$\Rightarrow \theta \equiv \left(1, \frac{1 - \cos\theta}{\sin\theta}\right)$$

$$\Rightarrow E \equiv \left(1, \frac{y\cos\theta}{\sin\theta}, y\right) \equiv (h, k) \qquad (\text{let})$$

$$\Rightarrow h = \left(\frac{1 - \text{Cos}\theta}{\text{Sin}\theta}\right) . \frac{\cos\theta}{\sin\theta}$$

$$k = \left(\frac{1 - \text{Cos}\theta}{\text{Sin}\theta}\right)$$



$$K = \frac{1 - \dfrac{h}{\sqrt{h^2 + K^2}}}{1 - \dfrac{K}{\sqrt{h^2 + K^2}}}$$

$$\Rightarrow K^2 + h = \sqrt{h^2 + K^2} \Rightarrow y^2 + x = \sqrt{x^2 + y^2}$$